Taylor-Maclaurin Series

Consider a function $f(x)$ defined by a power series of the form:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \qquad \cdots (1)$$

If we write out the expansion of $f(x)$ as:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

$$f(a) = c_0$$

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f'(a) = c_1$$

$$f''(x) = 2c_2 + 3\cdot 2c_3(x-a) + 4\times 3c_4(x-a)^2 + \cdots$$

$$f''(a) = 2c_2$$

$$f^{(3)}(x) = 3\cdot 2c_3 + 4\cdot 3\cdot 2c_4(x-a) + \cdots$$

$$f^{(3)}(a) = 3\cdot 2c_3$$

$\vdots$

$$f^{(n)}(x) = n!\, c_n$$

After computing the above derivatives, we observe that

$$f(a) = c_0, \quad f'(a) = c_1, \quad f''(a) = 2c_2 \;\Rightarrow\; c_2 = \frac{f''(a)}{2!} \text{ and } c_3 = \frac{f^{(3)}(a)}{3!}$$

In general, we have

$$\boxed{\boldsymbol{c_n = \frac{f^{(n)}(a)}{n!}}} \qquad \cdots (2)$$

Suppose that $f(x)$ has a power series expansion at $x = a$ then the series expansion of $f(x)$ takes the form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Which is called **Taylor Series**.

Maclaurin Series

If $a = 0$ in equation (1), then:

$$f(x) = \sum_{n=0}^{\infty} c_n\, x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots \qquad \cdots(3)$$

Which is called **Maclaurin Series**.

Taylor polynomial

The n^{th} partial sum of the Taylor series for a function f at a is known as the n^{th} -degree Taylor polynomial, denoted by $P_n(x)$. The 0^{th}, 1^{th}, 2^{th} and 3^{th} partial sum of the Taylor series are given by:

$$P_0(x) = f(a)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3$$

Example 1: Find $P_0(x), P_1(x), P_2(x)$ and $P_3(x)$ for $f(x) = \ln x$ at $x = 1$.

Solution: $f(x) = \ln x \quad ⇨ \quad f(1) = \ln 1 = 0$

$$f'(x) = \frac{1}{x} \quad ⇨ \quad f'(1) = 1$$

$$f''(x) = \frac{-1}{x^2} \quad ⇨ \quad f''(1) = -1$$

$$f^{(3)}(x) = \frac{2}{x^3} \quad ⇨ \quad f^{(3)}(1) = 2$$

$$P_0(x) = f(1) = 0$$

$$P_1(x) = f(1) + f'(1)(x-1) = (x-1)$$

$$P_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 = (x-1) - \frac{1}{2}(x-1)^2$$

$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f^{(3)}(1)}{3!}(x-1)^3$$

$$P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

Example 2: Compute the Maclaurin series of the following functions

1. $f(x) = e^{x}$ 2. $f(x) = e^{x^2}$

Solution: $f(x) = e^{x} \Rightarrow f(0) = e^{0} = 1$

$f'(x) = e^{x} \Rightarrow f'(0) = e^{0} = 1$

$f''(x) = e^{x} \Rightarrow f''(0) = e^{0} = 1$

$f^{(3)}(x) = e^{x} \Rightarrow f^{(3)}(0) = e^{0} = 1$

1. $e^{x} = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $e^{x^2} = \frac{1}{0!} + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

Example 3: Compute the Maclaurin series of the following functions:

1. $f(x) = \sin x$ 2. $f(x) = \frac{\sin(x^2)}{x^2}$

Solution: $f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f^{(3)}(x) = -\cos x \Rightarrow f^{(3)}(0) = -\cos 0 = -1$

We note that $f^{(2n+1)}(x) = (-1)^n \cos x \Rightarrow f^{(2n+1)}(0) = (-1)^n$

$f^{(2n)}(x) = (-1)^n \sin x \Rightarrow f^{(2n)}(0) = 0$

1. $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

2. $\sin(x^2) = \frac{(x^2)}{1!} - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$

$$\sin(x^2) = \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$\frac{\sin(x^2)}{x^2} = \frac{1}{1!} - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n+1)!}$$

Maclaurin Polynomials

The partial sums of Maclaurin series are called Maclaurin polynomials. More precisely, the Maclaurin polynomial of degree n of $f(x)$ (at $x = 0$) is the polynomial

$$P_n(x) = \sum_{n=0}^{n} \frac{f^{(n)}(0)}{n!} x^n$$

Example 4: Compute the Maclaurin polynomial of degree 3 for the function

$$f(x) = \cos x \ln(1-x)$$

Solution: Let $g(x) = \cos x$ and $h(x) = \ln(1-x)$

Maclaurin polynomial $P_3(x)$ of degree 3 of $f(x)$ is

$$P_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!} x^n = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3$$

$g(x) = \cos x \quad ⇨ \quad g(0) = 1$

$g'(x) = -\sin x \quad ⇨ \quad g'(0) = 0$

$g''(x) = -\cos x \quad ⇨ \quad g''(0) = -1$

$g'''(x) = \sin x \quad ⇨ \quad g'''(0) = 0$

$$g(x) = \cos x = 1 - \frac{x^2}{2!} = 1 - \frac{x^2}{2}$$

$h(x) = \ln(1-x) \quad ⇨ \quad h(0) = \ln(1) = 0$

$h'(x) = \frac{-1}{1-x} = -(1-x)^{-1} \quad ⇨ \quad h'(0) = -1$

$h''(x) = -(1-x)^{-2} \quad ⇨ \quad h''(0) = -1$

$h'''(x) = -2(1-x)^{-3} \quad ⇨ \quad h'''(0) = -2$

$$h(x) = \ln(1-x) = 0 - x - \frac{x^2}{2!} - \frac{2x^3}{3!} = -x - \frac{x^2}{2} - \frac{x^3}{3}$$

$$P_3(x) = \left(1 - \frac{x^2}{2}\right)\left(-x - \frac{x^2}{2} - \frac{x^3}{3}\right)$$

$$P_3(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} + \frac{x^3}{2} = -x - \frac{x^2}{2} + \frac{x^3}{6}$$

Example5 : Compute $P_3(x)$ of $f(x) = \dfrac{\ln(1+x)}{(1+x)}$

Solution: $g(x) = \ln(1+x) \quad ⇨ \quad g(0) = \ln(1) = 0$

$$g'(x) = \frac{1}{(1+x)} = (1+x)^{-1} \quad ⇨ \quad g'(0) = 1$$

$$g''(x) = -(1+x)^{-2} \quad ⇨ \quad g''(0) = -1$$

$$g'''(x) = 2(1+x)^{-3} \quad ⇨ \quad g'''(0) = 2$$

$$g(x) = \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

$$h(x) = \frac{1}{(1+x)} = (1+x)^{-1} \quad ⇨ \quad h(0) = 1$$

$$h'(x) = -(1+x)^{-2} \quad ⇨ \quad h'(0) = -1$$

$$h''(x) = 2(1+x)^{-2} \quad ⇨ \quad h''(0) = 2$$

$$h'''(x) = -6(1+x)^{-3} \quad ⇨ \quad h'''(0) = -6$$

$$h(x) = \frac{1}{(1+x)} = 1 - x + x^2 - x^3$$

$$f(x) = \frac{\ln(1+x)}{(1+x)} = \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)(1 - x + x^2 - x^3)$$

$$P_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3} - x^2 + \frac{x^3}{2} + x^3$$

$$P_3(x) = x - \frac{3x^2}{2} + \frac{11x^3}{6}$$

Exercises

1. Compute the Maclaurin series of the following functions:

   a. $f(x) = \dfrac{1 - \cos x}{x^2}$ b. $f(x) = \dfrac{x - \sin x}{x^2}$

2. Compute $P_3(x)$ of following functions:

   a. $f(x) = \sqrt{1+x}\ \cos x$ b. $f(x) = e^x \ln(1+x)$