

L'Hopital's rule

Suppose that we have one of the following cases,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{0}{0} \quad \text{or} \quad \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\mp\infty}{\mp\infty}$$

where c can be any real number, infinity or negative infinity. In these cases, we have,

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$$

Example: Evaluate the limits

$$1. \quad \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x^2} &= \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{e^x}{2} = \frac{1}{2} \end{aligned}$$

$$2. \quad \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x - \sin x} &= \lim_{x \rightarrow 0} \frac{2 \cos x - 2 \cos 2x}{1 - \cos x} = \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{-2 \sin x + 4 \sin 2x}{\sin x} = \frac{0}{0} \\ &= \lim_{x \rightarrow 0} \frac{-2 \cos x + 8 \cos 2x}{\cos x} = \frac{-2 + 8}{1} = 6 \end{aligned}$$

$$3. \quad \lim_{x \rightarrow \infty} \frac{5 + \ln 2x}{x^2 + 4} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{5 + \ln 2x}{x^2 + 4} = \lim_{x \rightarrow \infty} \frac{\frac{2}{2x}}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = \frac{1}{\infty} = 0$$

$$4. \quad \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{e^{2x}} = \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{e^{2x}} = \lim_{x \rightarrow \infty} \frac{(\ln x)^2}{(e^x)^2} = \left(\lim_{x \rightarrow \infty} \frac{\ln x}{e^x} \right)^2 = \left(\lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{e^x} \right)^2 = \left(\frac{0}{\infty} \right)^2 = 0$$

$$5. \lim_{x \rightarrow 0} \frac{x^2 e^{3x}}{(e^{2x} - 1)^2} = \frac{0}{0}$$

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x^2 e^{3x}}{(e^{2x} - 1)^2} &= \lim_{x \rightarrow 0} \frac{x^2}{(e^{2x} - 1)^2} \times \lim_{x \rightarrow 0} e^{3x} = \left(\lim_{x \rightarrow 0} \frac{x}{e^{2x} - 1} \right)^2 \times 1 \\ &= \left(\lim_{x \rightarrow 0} \frac{1}{2e^{2x}} \right)^2 = \left(\frac{1}{2} \right)^2 = \frac{1}{4} \end{aligned}$$

$$6. \lim_{x \rightarrow 0} \frac{\sin^{-1} 2x}{\tan^{-1} 3x} = \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{\sin^{-1} 2x}{\tan^{-1} 3x} = \lim_{x \rightarrow 0} \frac{2/\sqrt{1-4x^2}}{3/1+9x^2} = \frac{2}{3}$$

Exercises

Evaluate the limits

$$1. \lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{\cos x - 1}$$

$$2. \lim_{x \rightarrow 0} \frac{\cos 4x - \cos 2x}{x^2}$$

$$3. \lim_{x \rightarrow \infty} \frac{7 - \ln(2x+3)}{\ln(5x-2)}$$

$$4. \lim_{x \rightarrow -1} \frac{1 + \cos \pi x}{x^2 + 2x + 1}$$

$$5. \lim_{x \rightarrow 0} \frac{e^x - x - 1}{x \sin x}$$

$$6. \lim_{x \rightarrow 0} \frac{x^3}{x - \sin x}$$

$$7. \lim_{x \rightarrow 0} \frac{(1-x)^8 - 1}{2x}$$

$$8. \lim_{x \rightarrow \infty} \frac{\ln(100x)}{x^{0.01}}$$