The Differential Equations Variant Coefficients

A second order nonhomogeneous equation with variant coefficients is written as

$$P(x)y'' + Q(x)y' + R(x)y = F(x)$$

I- Euler's Differential Equation

We want to look for solutions to the differential equation in the form

$$ax^2y'' + bxy' + cy = f(x)$$

These types of differential equations are called *Euler equations.*

This differential equation can be transformed into a second order homogeneous equation with constant coefficients by use of the substitution: $x = e^t$.

Differentiating with respect to t gives:

$$\frac{dx}{dt} = e^t \quad \text{so} \quad \frac{dx}{dt} = x$$

In the Euler equation we see the term xy' which we need to substitute for, so we will multiply both sides of $\frac{dx}{dt} = x$ by y':

$$\frac{dy}{dx} \times \frac{dx}{dt} = x\frac{dy}{dx}$$

This simplifies to $\frac{dy}{dt} = xy'$.

Differentiating this with respect to x gives: $\frac{d}{dx}\frac{dy}{dt} = xy'' + y'$.

In the Euler equation we see the term xy' which we need to substitute for, so we will multiply the left side of the above equation by $\frac{dx}{dt}$ and the right-hand side by x (Remember they are equal) gives:

$$x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

So, under this substitution the Euler equation becomes:

$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = f(e^t)$, we can write it in the form:

$a\ddot{y} + (b-a)\dot{y} + cy = f(e^t); \quad \ddot{y} = \frac{d^2y}{dt^2}, \dot{y} = \frac{dy}{dt}$

This is a second order linear equation with constant coefficients.

Example 1: Solve $x^2y'' + xy' + y = 2$

Solution: The given differential equation is Euler equation

Put $x = e^t$

Then we get $\ddot{y} + (1-1)\dot{y} + y = 2$

Or $\ddot{y} + y = 2$

$$m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$y_h = c_1 \sin t + c_2 \cos t$$

Let $y_p = A$ then $\dot{y} = \ddot{y} = 0 \Rightarrow A = 2 \Rightarrow y_p = 2$

$$y = y_h + y_p \equiv c_1 \sin t + c_2 \cos t \underline{+ 2}$$

$$y = y_h + y_p \equiv c_1 \sin t + c_2 \cos t \underline{+ 2}$$

We have $x = e^t \Rightarrow t = \ln x$

Then $y = c_1 \sin(\ln x) + c_2 \cos(\ln x) \underline{+ 2}$

Example 2: Solve $x^2y'' - 2xy' + 2y = 4x^3$

Solution: The given differential equation is Euler equation

Put $x = e^t$

Then we get $\ddot{y} + (-2-1)\dot{y} + 2y = 4e^{3t}$

$\ddot{y} - 3\dot{y} + 2y = 4e^{3t}$

$m^2 - 3m + 2 = 0 \Rightarrow (m-1)(m-2) = 0 \Rightarrow m_1 = 1$ and $m_2 = 2$

$$y_h = c_1e^t + c_2e^{2t}$$

Let $y_p = Ae^{3t}$ then $\dot{y} = 3Ae^{3t}$ and $\ddot{y} = 9Ae^{3t}$

$9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} = 4e^{3t} \Rightarrow A = 2 \Rightarrow y_p = 2e^{3t}$

So $y = y_h + y_p = c_1e^{2t} + c_2e^t + 2e^{3t}$

Then $y = c_1x^2 + c_2x + 2x^3$

Example 3: Solve $x^2y'' + 6xy' + 6y = \ln x$

Solution: The given differential equation is Euler equation

Put $x = e^t$

Then we get $\ddot{y} + (6-1)\dot{y} + 6y = t$

$\ddot{y} + 5\dot{y} + 6y = t$

$$m^2 + 5m + 6 = 0 \Rightarrow (m+3)(m+2) = 0 \quad \Rightarrow \quad m_1 = -3 \quad , m_2 = -2$$

$$y_h = c_1e^{-3t} + c_2e^{-2t}$$

Let $y_p = At + B$ then $\dot{y} = A$ and $\ddot{y} = 0$

$$5A + 6At + 6B = t$$

$$6A = 1 \Rightarrow A = (1/6)$$

$$5A + 6B = 0 \quad \Rightarrow \quad B = -(5/36)$$

$$y_p = (1/6)t - (5/36)$$

$$y = y_h + y_p = c_1e^{-3t} + c_2e^{-2t} + (1/6)t - (5/36)$$

$$y = c_1x^{-3} + c_2x^{-2} + (1/6)\ln x - (5/36)$$

II- Power Series Method

We consider the second-order linear homogeneous differential equation for $y = y(x)$

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

where $P(x)$, $Q(x)$ and $R(x)$ are polynomials. The idea of power series method is to assume that the unknown function y can be expanded into a power series:

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$$

Substituting in the differential equation and requiring that the coefficients of each power of x sum to zero, we can find all the constants $a_n \ \forall\, n = 2,3,4,\cdots$ in terms of a_0 or a_1.

Example 4: Solve $y'' + 2xy' + 2y = 0$ by power series method

Solution: Let $y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = \mathbf{2a_2} + \mathbf{6a_3} x + \mathbf{12a_4} x^2 + \mathbf{20a_5} x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$$

$$2xy' = \mathbf{0} + \mathbf{2a_1} x + \mathbf{4a_2} x^2 + \mathbf{6a_3} x^3 + 8a_4 x^4 + 12a_5 x^5 + \cdots$$

$$2y = \mathbf{2a_0} + \mathbf{2a_1} x + \mathbf{2a_2} x^2 + \mathbf{2a_3} x^3 + 2a_4 x^4 + 2a_5 x^5 + \cdots$$

Now, sum of the null terms of x equal to zero: $2a_2 + 2a_0 = 0 \Rightarrow \boxed{a_2 =- a_0}$

Sum of coefficients of x equal to zero; $6a_3 + 2a_1 + 2a_1 = 0 \Rightarrow \boxed{a_3 =- (2/3)a_1}$

Sum of coefficients of x^2 equal to zero:

$12a_4 + 4a_2 + 2a_2 = 0 \Rightarrow a_4 =- (1/2)a_2 \Rightarrow \boxed{a_4 = (1/2)a_0}$

Sum of coefficients of x^3 equal to zero:

$20a_5 + 6a_3 + 2a_3 = 0 \Rightarrow a_5 =- (2/5)a_3 \Rightarrow a_5 = (4/15)a_1$

$y = a_0 + a_1 x - a_0 x^2 - (2/3)a_1 x^3 + (1/2)a_0 x^4 + (4/15)a_1 x^5 + \cdots$

$$y = a_0\left(1 - x^2 + \frac{1}{2}x^4 + \cdots\right) + a_1\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 + \cdots\right)$$

Example 5: Solve $y'' - xy' + 4y = 0$ by using power series method

Solution :

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$y'' = \mathbf{2a_2} + \mathbf{6a_3}x + \mathbf{12a_4}x^2 + \mathbf{20a_5}x^3 + \mathbf{30a_6}x^4 + 42a_7x^5 + \cdots$$

$$-xy' = 0 \quad - \mathbf{a_1}\ x - \mathbf{2a_2}\ x^2 - \mathbf{3a_3}\ x^3 - \mathbf{4a_4}\ x^4 - 5a_5\ x^5 - \cdots$$

$$4y = \mathbf{4a_0} + \mathbf{4a_1}\ x + \mathbf{4a_2}\ x^2 + \mathbf{4a_3}\ x^3 + \mathbf{4a_4}\ x^4 + 4a_5\ x^5 + \cdots$$

$2a_2 + 4a_0 = 0 \Rightarrow \boxed{a_2 = -2a_0}$, $6a_3 - a_1 + 4a_1 = 0 \Rightarrow \boxed{a_3 = -(1/2)a_1}$

$12a_4 - 2a_2 + 4a_2 = 0 \Rightarrow a_4 = -(1/6)a_2 \Rightarrow \boxed{a_4 = (1/3)a_0}$

$20a_5 - 3a_3 + 4a_3 = 0 \Rightarrow a_5 = -(1/20)a_3 \Rightarrow \boxed{a_5 = (1/40)a_1}$

$30a_6 - 4a_4 + 4a_4 = 0 \Rightarrow \boxed{a_6 = 0}$ and $a_8 = a_{10} = \cdots = 0$

$$y = a_0 + a_1x - 2a_0x^2 - (1/2)a_1x^3 + (1/3)a_0x^4 + (1/40)a_1x^5 + \cdots$$

$$y = a_0\left(1 - 2x^2 + \frac{x^4}{3}\right) + a_1\left(x - \frac{x^3}{2} + \frac{x^5}{40} + \cdots\right)$$

Example 6: Solve $(x^2+1)\,y'' - 4xy' + 6y = 0$ by using power series method

Solution :

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$$

$$y' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + 6a_6x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots$$

$$6y = 6a_0 + 6a_1x + 6a_2x^2 + 6a_3x^3 + 6a_4x^4 + 6a_5x^5 + \cdots$$

$$-4xy' = 0 - 4a_1x - 8a_2x^2 - 12a_3x^3 - 16a_4x^4 - 20a_5x^5 - \cdots$$

$$y'' = 2a_2 + 6a_3x + 12a_4x^2 + 20a_5x^3 + 30a_6x^4 + 42a_7x^5 + \cdots$$

$$x^2y'' = 0 + 0 + 2a_2x^2 + 6a_3x^3 + 12a_4x^4 + 20a_5x^5 + \cdots$$

$6a_0 + 2a_2 = 0 \quad ⇨ \quad \boxed{a_2 = -3a_0}$

$6a_1 - 4a_1 + 6a_3 = 0 \quad ⇨ \quad a_3 = -1/3\ a_1$

$6a_2 - 8a_2 + 12a_4 + 2a_2 = 0 \quad ⇨ \quad 12a_4 = 0 \quad ⇨ \quad a_4 = 0$

$6a_3 - 12a_3 + 20a_5 + 6a_3 = 0 \quad ⇨ \quad 20a_5 = 0 \quad ⇨ \quad a_5 = 0$

$y = a_0 + a_1x - 3a_0x^2 - 1/3\ a_1x^3$

$$y = a_0(1-3x^2) + a_1\left(x - \frac{x^3}{3}\right)$$

Exercises

Solve Euler differential equations

(1) $x^2y'' - 5xy' + 8y = 2x^2$ (2) $x^2y'' - xy' - 3y = x^5$

(3) $x^2y'' + 5xy' + 4y = 1/x^2$ (4) $x^2y'' + 3xy' + 5y = \ln x$

Solve the differential equations by using power series method

(5) $x\,y'' + y' + xy = 0$ (6) $x^2y'' + y' + x^2y = 0$

(7) $y'' + 3xy' - y = 0$ (8) $(x-1)y'' + 2y' = 0$