The Simplest Type of PDE

Some initial value problems can be solved by using antiderivatives partial.

The following examples explain how to find the solution of PDE.

Example 1: Solve the initial value problem $u_x(x,y) = 2ye^{2x}$ with $u(0,y) = e^{-2y}$

Solution: $\int u_x(x,y)\,dx = \int 2ye^{2x}dx \;\Rightarrow\; u(x,y) = ye^{2x} + f(y)$

To find $f(y)$: $u(0,y) = e^{-2y} \;\Rightarrow\; e^{-2y} = y + f(y)$

$\therefore\; f(y) = e^{-2y} - y$

The solution of PDE is: $u(x,y) = ye^{2x} + e^{-2y} - y$

Example 2: Solve the initial value problem $u_t(x,t) = \cos t$ with $u(x,0) = \sin x$

Solution: $\int u_t(x,t)\,dt = \int \cos t\,dt \;\Rightarrow\; u(x,t) = \sin t + f(x)$

To find $f(x)$: $u(x,0) = \sin x \;\Rightarrow\; \sin x = 0 + f(x)$

$\therefore\; f(x) = \sin x$

The solution of PDE is: $u(x,t) = \sin t + \sin x$

Example 3: Solve the initial value problem $u_{xx}(x,y) = 0\,;\; 0 \le x \le 1$

with $u(0,y) = y^2$ and $u(1,y) = 1$

Solution: $\int u_{xx}(x,y)dx = \int 0dx \;\Rightarrow\; u_x(x,y) = f(y)$

$\int u_x(x,y)\,dx = \int f(y)dx \;\Rightarrow\; u(x,y) = xf(y) + h(y)$

To find $h(y)$: $u(0,y) = y^2 \;\Rightarrow\; 0 \times f(y) + h(y) = y^2$

$\therefore\; h(y) = y^2$

To find $f(y)$: $u(1,y) = 1 \;\Rightarrow\; 1 \times f(y) + y^2 = 1$

$\therefore\; f(y) = 1 - y^2$

The solution of PDE is: $u(x,y) = x(1 - y^2) + y^2$

Example 4: Solve the initial value problem $u_{xx}(x,y) = 6xy$

with $u(1,y) = \ln(y^2+1)$ and $u_x(1,y) = 0$

Solution: $\int u_{xx}(x,y)dx = \int 6xy\,dx \;\Rightarrow\; u_x(x,y) = 3x^2y + f(y)$

To find $f(y)$: $u_x(1,y) = 0 \;\Rightarrow\; 0 = 3y + f(y)$

$\therefore\; f(y) = -3y$

So, $u_x(x,y) = 3x^2y - 3y$

$\int u_x(x,y)\,dx = \int (3x^2y - 3y)\,dx$

$u(x,y) = x^3y - 3xy + h(y)$

To find $h(y)$: $u(1,y) = \ln(y^2+1) \;\Rightarrow\; \ln(y^2+1) = y - 3y + h(y)$

$\therefore h(y) = 2y + \ln(y^2+1)$

The solution of PDE is: $u(x,y) = x^3y - 3xy + 2y + \ln(y^2+1)$

Example 5: Solve the initial value problem $u_{xy}(x,y) = 2x$, $u(0,y) = 0$, $u(x,0) = x^2$

Solution: $\int u_{xy}(x,y)dx = \int 2x\,dx \;\Rightarrow\; u_y(x,y) = x^2 + f(y)$

$\int u_y(x,y)dy = \int \left(x^2 + f(y)\right)dy \;\Rightarrow\; u(x,y) = x^2y + F(y) + h(x)$

To find $F(y)$ and $h(x)$: $u(0,y) = 0 \;\Rightarrow\; 0 = 0 + f(y) + h(0)$

$F(y) = -h(0)$ $\cdots(1)$

And $u(x,0) = x^2 \;\Rightarrow\; x^2 = 0 + F(0) + h(x)$

$h(x) = x^2 - F(0)$ $\cdots(2)$

So, $u(x,y) = x^2y - h(0) + x^2 - F(0)$

Or $u(x,y) = x^2y + x^2 - \left(h(0) + F(0)\right)$

To find the value of $h(0) + F(0)$, put $y = 0$ in equation (1)

We get $F(0) = -h(0) \;\Rightarrow\; F(0) + h(0) = 0$

The solution of PDE is: $u(x,y) = x^2y + x^2$

Example 6: Solve the initial value problem $u_{xy}(x,y) = 6x^2y$

with $u(1,y) = \cos y$ and $u(x,0) = x^2$

Solution: $\int u_{xy}(x,y)dx = \int 6x^2y\,dx \;\Rightarrow\; u_y(x,y) = 2x^3y + f(y)$

$$\int u_y(x,y)dy = \int \left(2x^3y + f(y)\right) dy$$

$$u(x,y) = x^3y^2 + F(y) + h(x)$$

To find $F(y)$ and $h(x)$: $u(1,y) = \cos y \;\Rightarrow\; \cos y = y^2 + F(y) + h(1)$

$$F(y) = \cos y - \; y^2 - h(1) \qquad \cdots(1)$$

And $u(x,0) = x^2 \;\Rightarrow\; x^2 = 0 + F(0) + h(x)$

$$h(x) = x^2 - F(0) \qquad \cdots(2)$$

$$u(x,y) = x^3y^2 + \cos y - y^2 - h(1) + x^2 - F(0)$$

$$u(x,y) = x^3y^2 + \cos y - y^2 + x^2 - \left(h(1) + F(0)\right)$$

To find the value of $h(1) + F(0)$: put $x = 1$ in equation (2)

We get $h(1) = 1 - F(0) \;\Rightarrow\; F(0) + h(1) = 1$

The solution of PDE is: $u(x,y) = x^3y^2 + \cos y - y^2 + x^2 - 1$

H.W: Solve the initial value problems

1. $u_y(x,y) = \tan y$ with $u(x,0) = \ln x$
2. $u_{xx}(x,y) = 12x + 4y$ with $u_x(1,y) = y^3 + 6$ and $u(0,y) = \cos y$
3. $u_{xx}(x,y) = y^2e^{xy}$ with $u_x(0,y) = \cos y$ and $u(0,y) = y + 2$
4. $u_{xy}(x,y) = -\sin(x-y) - \cos(x+y)$

   with $u(0,y) = \cos y - \sin y$ and $u_x(x,0) = \cos x$
5. $u_{yy}(x,y) = \sin(x - 2y)$ with $u(x,\pi) = \frac{3}{4}\sin x$, $u_y(x,\pi) = -\frac{1}{2}\cos x$