## Multiple Integrals

The multiple integral is a generalization of the definite integral to functions of more than one real variable. Integrals of a function of two variables are called double integrals, and integrals of a function of three variables are called triple integrals.

## Double Integrals

The expression $\int_c^d \int_a^b f(x,y)dxdy$ is called double integral and indicates that:

1. $f(x,y)$ is first integrated with respect to $x$ (regarding $y$ as being constant) between the limits $x = a$ and $x = b$.

2. the result is then integrated with respect to $y$ between the limits $y = c$ and $y = d$.

Example 1: Evaluate $\int_1^2 \int_2^4 (x + 2y)dxdy$

$$\int_1^2 \int_2^4 (x + 2y)dxdy = \int_1^2 \frac{x^2}{2} + 2yx\Big|_2^4 dy = \int_1^2 (8 + 8y - 2 - 4y)\, dy$$

$$= \int_1^2 (4y + 6)\, dy = 2y^2 + 6y\Big|_1^2 = 8 + 12 - 2 - 6 = 12$$

Example 2: Evaluate $\int_0^{\pi} \int_0^{\sin x} y\, dydx$

$$\int_0^{\pi} \int_0^{\sin x} y\, dydx = \int_0^{\pi} \frac{y^2}{2}\Big|_0^{\sin x} dx = \int_0^{\pi} \frac{\sin^2 x}{2} dx = \int_0^{\pi} \frac{1 - \cos 2x}{4} dx$$

$$= \frac{1}{4}x - \frac{1}{8}\sin 2x\Big|_0^{\pi} = \frac{\pi}{4}$$

Example 3: Evaluate $\int_1^2 \int_y^{y^2} dxdy$

$$\int_1^2 \int_y^{y^2} dxdy = \int_1^2 x\Big|_y^{y^2} dy = \int_1^2 (y^2 - y)\, dy = \frac{y^3}{3} - \frac{y^2}{2}\Big|_1^2 = \frac{5}{6}$$

Example 4: Evaluate $\int_0^1 \int_0^x \sqrt{1-x^2}\, dydx$

$$\int_0^1 \int_0^x \sqrt{1-x^2}\, dydx = \int_0^1 y\Big|_0^x \sqrt{1-x^2} dx = \int_0^1 x\sqrt{1-x^2}\, dx = \frac{-1}{2}\int_0^1 -2x(1-x^2)^{1/2}\, dx$$

$$= -\frac{1}{2}(1-x^2)^{3/2} \times \frac{2}{3}\Big|_0^1 = -\frac{1}{3}(0-1) = \frac{1}{3}$$

## Triple Integrals

Triple Integral is one of the types of multi integral of a function that involves three variables. Triple Integral in Calculus is the integration involving volume; hence it is also called Volume Integral and the process of calculating Triple Integral is called Triple Integration.

Let's do a quick example of this type of triple integral.

Example 5: Evaluate $\int_0^2 \int_{-1}^1 \int_0^1 (2x - y + z) dxdydz$

$$\int_0^2 \int_{-1}^1 \int_0^1 (2x-y+z)dxdydz = \int_0^2 \int_{-1}^1 (x^2 - yx + zx)\Big|_0^1 dydz = \int_0^2 \int_{-1}^1 (1-y+z)dydz$$

$$= \int_0^2 \left(y - \frac{y^2}{2} + zy\right)\Big|_{-1}^1 dz = \int_0^2 \left(1 - \frac{1}{2} + z - \left(-1 - \frac{1}{2} - z\right)\right) dz$$

$$= \int_0^2 (2+2z)dz = 2z + z^2 \Big|_0^2 = 8$$

Example 6: Evaluate $\int_0^{\pi}\int_0^{\pi}\int_0^{3} x^2 \sin\theta \, dx d\theta d\emptyset$

$$\int_0^{\pi}\int_0^{\pi}\int_0^{3} x^2 \sin\theta \, dx d\theta d\emptyset = \int_0^{\pi}\int_0^{\pi}\left(\frac{x^3}{3}\Big|_0^3\right)\sin\theta \, d\theta d\emptyset = \int_0^{\pi}\int_0^{\pi} 9\sin\theta \, d\theta d\emptyset$$

$$= \int_0^{\pi} -9\cos\theta\Big|_0^{\pi} d\emptyset = \int_0^{\pi} -9(-1-1)d\emptyset = \int_0^{\pi} 18 d\emptyset = 18\emptyset\Big|_0^{\pi} = 18\pi$$

## Exercises

Evaluate each of the following integrals

1. $\int_0^2\int_1^{e^x} dydx$
2. $\int_0^1\int_{\sqrt{y}}^1 dxdy$
3. $\int_{-2}^1\int_{x^2+4x}^{3x+2} dydx$
4. $\int_0^1\int_{\sqrt{y}}^{2-\sqrt{y}} xydxdy$
5. $\int_0^1\int_{x^2}^1 (x+y)dydx$
6. $\int_0^1\int_0^{y^2} \sqrt{y^3+3}\, dxdy$
7. $\int_0^1\int_0^2 x\sqrt{4-x^2}\, dxdy$
8. $\int_0^2\int_1^3\int_1^2 xy^2zdxdydz$
9. $\int_0^2\int_0^1\int_0^1 (x^2+y^2+z^2)dxdydz$
10. $\int_0^2\int_{3x/2}^{3}\int_0^{5x/2} dzdydx$