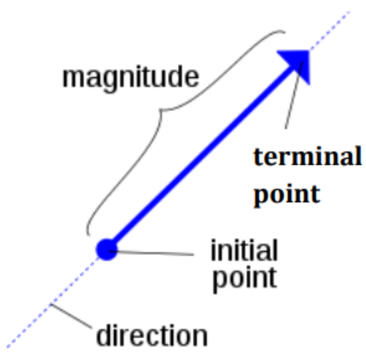
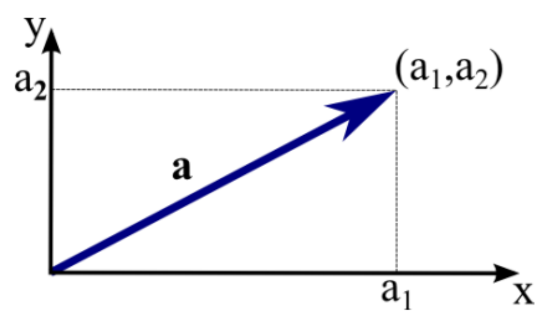
Vectors

1. Vectors in Two Dimensions

In order to distinguish vectors from scalar, we will se hod-faced letters, to denote vectors; for example, **a, b, U**, and so forth. Vectors have both, magnitude and direction, while scalars have only magnitude. For that, we will represent vectors by arrows. The magnitude of the vector **a** is the length of the arrow, and its direction is the direction of the arrow.





When the vectors intial point is placed at the origin, we write:

$$\boldsymbol{a} = (a_1, a_2)$$

to denote the vector from O to R.

Length of a:

The magnitude of a is the distance from O to R, and is denoted by $\|\boldsymbol{a}\|$:

$$\|\boldsymbol{a}\| = \sqrt{{a_1}^2 + {a_2}^2}$$

Equality:

The vectors $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ are equal if and only if

$$a_1 = b_1 \; and \; a_2 = b_2$$

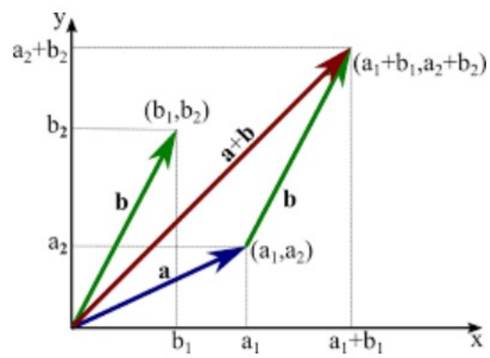
Zero:

The zero vector is the vector of length zero. As a result, if its intial point is at the origin, then so its terminal point, so that

$$\mathbf{0} = (0,0)$$

Addition:

If the intial point of **b** is placed at the terminal point of **a**, then **a+b** is the vector drawn from the intial point of **a** to the terminal point of **b**.



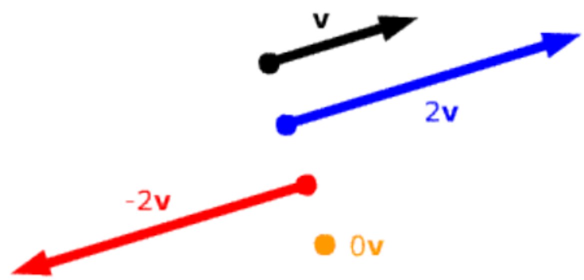
$$\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2)$$

Multiplication of a vector by a scalar:

Let **a** be a vector, and γ be a scalar (real number). If $\gamma > 0$, then $\gamma\boldsymbol{a}$ or $\boldsymbol{a}\gamma$ is the vector whose direction is the same as that of **a** and whose length id γ times the length of **a**.

If $\gamma < 0$, then the direction of $\gamma \boldsymbol{a}$ is opposite to that of **a** and whose length is $|\gamma|$ times the length of **a**.

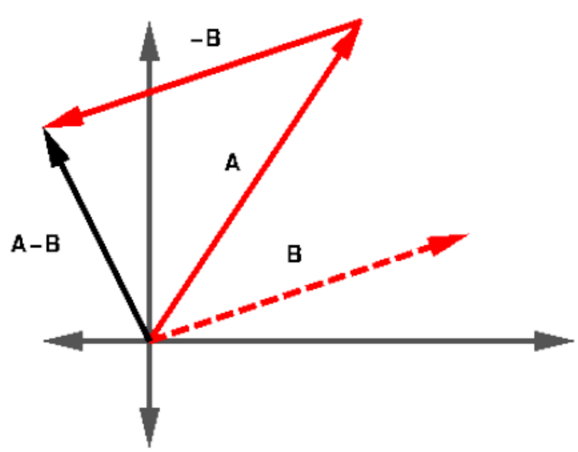
If $\gamma = 0$, or **a**=0, then $\gamma \boldsymbol{a}$ is the zero vector.



$$\gamma \boldsymbol{a} = (\gamma a_1, \gamma a_2)$$

Subtraction:

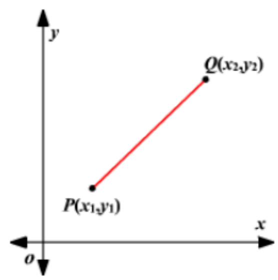
The vector $\boldsymbol{a} - \boldsymbol{b}$ is drawn from the terminal point of **b** to the terminal point of **a.**



$$\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2)$$

Finding Vectors using their end points:

The vector whose intial point is $P(p_1, p_2)$ and terminal point is $Q(q_1, q_2)$ denoted by $\overrightarrow{PQ}$, can be found by subtracting the coordinates of its intial point from the corresponding coordinates of its terminal point:



$$\overrightarrow{PQ} = (q_1, q_2) - (p_1, p_2) = (q_1 - p_1, q_2 - p_2)$$

The Dot Product:

we have two ways to multiply a vector by another. The first type is dot product which gives a scalar as a result of multiply two vectors together.

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_1 \cdot b_1 + a_2 \cdot b_2$$

Relation between dot product and length of a vector:

When we multiply a vector by itself using dot product, we get

$$\boldsymbol{a} \cdot \boldsymbol{a} = {a_1}^2 + {a_2}^2 = \|a\|^2$$

Also, if γ is a scalar, then $\|\gamma \boldsymbol{a}\| = \|\gamma a_1, \gamma a_2\| = \sqrt{\gamma^2 {a_1}^2 + \gamma^2 {a_2}^2} = |\gamma|\sqrt{{a_1}^2 + {a_2}^2} = |\gamma|\|a\|$

Unit Vectors:

Any vector of length one is called a unit vector. To determine a unit vector of length one in the direction of a vector **a** we use:

$$\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$$

Geometrical Interpolation of Dot Product:

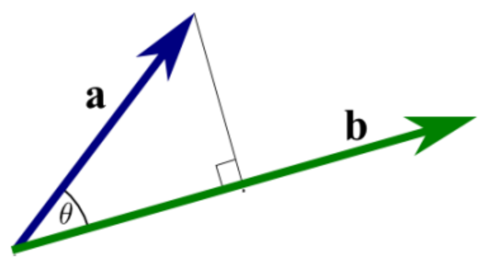
Let **a** and **b** be nonzero vectors which have the same initial point. And let θ be the angle between them s.t. $0 \le \theta \le \pi$. Use the law of cosines from trigonometry for the triangle below:

$$\|\boldsymbol{a}-\boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\|\boldsymbol{a}\|\|\boldsymbol{b}\|cos\theta$$

We obtain

$$\boldsymbol{a}\cdot\boldsymbol{b} = \|\boldsymbol{a}\|\|\boldsymbol{b}\|cos\theta$$

We can use the relation above to find the angle between vectors as follow:



$$cos\theta = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\|\boldsymbol{a}\|\|\boldsymbol{b}\|} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}\cdot\frac{\boldsymbol{b}}{\|\boldsymbol{b}\|} = \boldsymbol{u}\cdot\boldsymbol{v}$$

where $\boldsymbol{u}$ and $\boldsymbol{v}$ are unit vectors of $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively.

Orthogonal Vectors:

The vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal if and only if $\boldsymbol{a} \cdot \boldsymbol{b} = \mathbf{0}$

Unit Coordinate Vectors:

Unit vectors in the direction of x-axis and y-axis are important enough so that the special symbols $\boldsymbol{i}$ and $\boldsymbol{j}$ are reversed for them. Their components are

$$\boldsymbol{i} = (1,0)\ , \boldsymbol{j} = (0,1)$$

Also, it is readily to verify that

$$\boldsymbol{i}.\boldsymbol{i} = 1, \qquad \boldsymbol{j}.\boldsymbol{j} = 1, \qquad \boldsymbol{i}.\boldsymbol{j} = \boldsymbol{j}.\boldsymbol{i} = 0$$

Q: Prove that **i** and **j** are orthogonal?

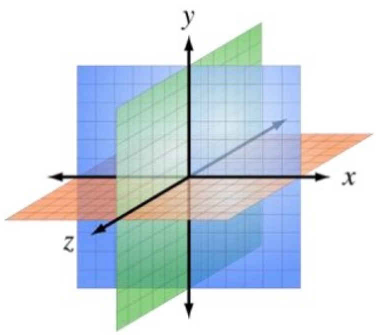
We can express a given vector $\boldsymbol{a} = (a_1, a_2)$ in terms of **i** and **j**, we have

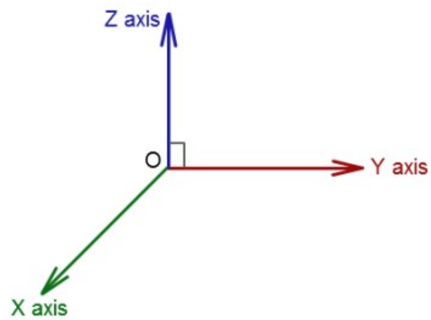
$$\boldsymbol{a} = (a_1, a_2) = a_1\mathbf{i} + a_2\,\mathbf{j}$$

Triangle Inequality:

$$\|\boldsymbol{a} + \boldsymbol{b}\| \leq \|\boldsymbol{a}\| + \|\boldsymbol{b}\|$$

2. Vectors in Three Dimensions





A three-dimensional vector **a** may be thought of as an arrow in three-dimensional xyz-space. Let us place the initial point at the origin, then the terminal point coincides with some point P. If $P(a_1, a_2, a_3)$, then we write

$$\boldsymbol{a} = (a_1, a_2, a_3),$$

The length of a: $\|\boldsymbol{a}\| = \sqrt{{a_1}^2 + {a_2}^2 + {a_3}^2}$

Equality: The vectors $\boldsymbol{a} = \boldsymbol{b}$ if and only if $a_1 = b_1, a_2 = b_2 and\ a_3 = b_3$

Zero: The zero vector is $\mathbf{0} = (0,0,0)$

Addition: $\boldsymbol{a} + \boldsymbol{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$

Multiplication of a vector by a scalar: $\gamma\boldsymbol{a} = (\gamma a_1, \gamma a_2, \gamma a_3)$

Subtraction: $\boldsymbol{a} - \boldsymbol{b} = (a_1 - b_1, a_2 - b_2, a_3 - b_3)$

Dot Product: $\boldsymbol{a} \cdot \boldsymbol{b} = a_1 \cdot b_1 + a_2 \cdot b_2 + a_3 \cdot b_3$

Unit Vectors: $\boldsymbol{u} = \frac{\boldsymbol{a}}{\|\boldsymbol{a}\|}$

There are now three unit coordinate vectors, denoted by

$$\boldsymbol{i} = (1,0,0)\ ,\boldsymbol{j} = (0,1,0)\ ,\boldsymbol{k} = (0,0,1)$$

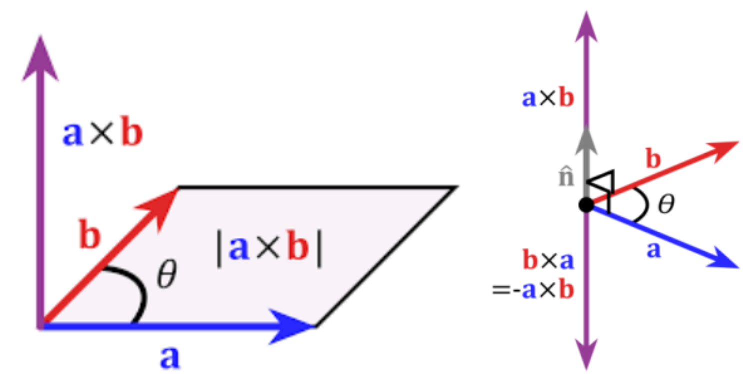
In terms of these vectors, we can write the vector $\boldsymbol{a} = (a_1, a_2, a_3)$ as

$$\boldsymbol{a} = a_1\boldsymbol{i} + a_2\boldsymbol{j} + a_3\boldsymbol{k},$$

Cross Product:

In dot Product, we get a scalar as a result of multiplication. In cross product, we multiply to vectors and get another vector as a result.

Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be nonzero vectors, and θ be the angle between them, s.t. $0 \leq \theta \leq \pi$. Then $\boldsymbol{a} \times \boldsymbol{b}$ is defined to be the vector with the following properties:



1. $\boldsymbol{a} \times \boldsymbol{b}$ is perpendicular at to both **a** and **b**
2. The magnitude of $\boldsymbol{a} \times \boldsymbol{b}$ is given by

$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\|\|\boldsymbol{b}\| sin\theta$$

3. The direction of $\boldsymbol{a} \times \boldsymbol{b}$ is chosen so that when a is rotated into b through the angle θ, then **a, b**, $\boldsymbol{a} \times \boldsymbol{b}$ form a right handed system of vectors.