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Chapter Five: Dimensional Analysis

DA1

Dimensional Analysis

This concept is useful to:

1. check units to any equation
2. Express experimental data easily.
3. Reduce the variables and then reduce the number of tests
4. help in physical modeling

For example: Coefficient of drag (C_D) that exert by a certain fluid flow on a body can be calculate by adopting the following variables: ρ, V, d, μ and ϵ. By using dimensional analysis, we can express C_D as: $C_D = f(Re, \frac{\epsilon}{D})$

Primary Dimensions (MLTθ system)

It is easy to express or convert any physical quantity by converting its known unit to this system as in the following:

kg $\longrightarrow$ [M]
m $\longrightarrow$ [L]
sec $\longrightarrow$ [T]
°k $\longrightarrow$ [θ]

DA2

Ex:- Convert the following quantities to MLTθ - system or (primary dimensions)?

① area ② force ③ power ④ viscosity
⑤ flow rate ⑥ velocity ⑦ Density

Sol.

① Area $= m^2 = [L]^2$

② force $= N = kg \cdot \frac{m}{s^2} = M \cdot \frac{L}{T^2} = \left[\frac{M \cdot L}{T^2}\right]$

③ power $= \frac{J}{s} = \frac{N \cdot m}{s} = \frac{kg \cdot \frac{m}{s^2} \cdot m}{s} = \frac{kg \cdot m^2}{s^3} = \left[\frac{ML^2}{T^3}\right]$

④ viscosity $(\mu) = Pa \cdot s = \frac{N}{m^2} \cdot s = \frac{kg \cdot \frac{m}{s^2}}{m^2} \cdot s = \frac{kg}{m \cdot s} = \left[\frac{M}{L \cdot T}\right]$

⑤ flow rate $(Q) = \frac{m^3}{s} = \left[\frac{L^3}{T}\right]$

⑥ velocity $(V) = \frac{m}{s} = \left[\frac{L}{T}\right]$

⑦ Density $\rho = \frac{kg}{m^3} = \left[\frac{M}{L^3}\right]$

Dimensional Homogeneity

All equations must have same units in their sides to be correct equations

Ex:- check Bernoulli equation for dimensional homogeneity

Sol.

B.E. $\frac{P_1}{\gamma} + \frac{V_1^2}{2g} + Z_1 = \frac{P_2}{\gamma} + \frac{V_2^2}{2g} + Z_2$

$\frac{P_1}{\gamma} = \frac{\frac{N}{m^2}}{\frac{N}{m^3}} = m = [L]$

$\frac{V_1^2}{2g} = \frac{\frac{m^2}{s^2}}{2 \cdot \frac{m}{s^2}} = m = [L]$

$Z = m = [L]$

L.H.S is [L] and same for R.H.S

∴ B.E is dimensional homogenous.

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Ex2 : Check $h_f = f \frac{L}{D} \frac{V^2}{2g}$ for dimensional homogeneity?

Sol.

$h_f = m = [L]$ (L.H.S)

f Dimensionless

$L = m = [L]$

$D = m = [L]$

$V^2 = \frac{m^2}{s^2} = \left[\frac{L^2}{T^2}\right]$

$g = \frac{m}{s^2} = \left[\frac{L}{T^2}\right]$

2 dimensionless

$\Rightarrow f \frac{L}{D} \frac{V^2}{2g} \Rightarrow \frac{L}{D} \frac{V^2}{g} = \frac{[L]}{[L]} \cdot \frac{\frac{L^2}{T^2}}{\frac{L}{T^2}} = [L]$ (R.H.S)

$\therefore [L] = [L]$

$\therefore$ the equation is $\boxed{\text{homogenous}}$

Buckingham's π – Theorem

This is a way to combined two or more variables into some dimensionless groups, so that, number of tests could be reduced.

Ex:- C_D is depending on ρ V d μ of a ball set up inside a certain fluid flow. Now to get the relationship. we need to make 10^4 tests if we consider 10 tests for each variable. If each test costs 30 minutes to be done, then 10^4 test needs about 2 years. By using this theorem, we can write $C_D = f(Re)$ and we need

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to few hours to get the correlation.

Steps of Solution

① Identify the involving variables (n)

② Identify the primary dimensions (M L T θ)

③ Write the primary dimensions of each variable

④ Get n and m

n :- whole variables

m :- repeating variables (ρ v d always)

⑤ get π – group number (n−m) and get the groups

$$\pi_1 = \text{non-primary variable} * \rho^a \, v^b \, d^c$$

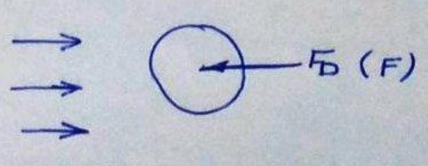
⑥ get the values a, b & c by equating π_1 to $M^0 L^0 T^0$

⑦ write π – relations as $\pi_1 = f(\pi_2, \pi_3, --- \pi_{n-m})$

Ex1 Get an expression of drag force applied by a certain fluid flow on a spherical ball by using dimensional analysis concept ?

$$C_D = \frac{\tau}{\frac{1}{2}\rho v^2} = \frac{F_D/A}{\frac{1}{2}\rho v^2}$$

Sol. ① $F = f(\rho, v, d, \mu)$



F ρ v d μ

$F_D =$

DAS

② no heat $\therefore$ primary dimensions M L T only

③ Primary dimensions for each variable

F	ρ	V	d	μ
$\frac{ML}{T^2}$	$\frac{M}{L^3}$	$\frac{L}{T}$	L	$\frac{M}{L.T}$

$F \rightarrow N$

④ $n = 5$ $m = 3$ $\therefore n - m = 2$

$\boxed{2 \text{ groups}}$

⑤
$$\pi_1 = F \rho^a V^b d^c$$
$$\pi_2 = \mu \rho^a V^b d^c$$

⑥
$$\pi_1 = M^0 L^0 T^0 = \left[\frac{ML}{T^2}\right]\left[\frac{M}{L^3}\right]^a\left[\frac{L}{T}\right]^b\left[L\right]^c$$

for M: $0 = 1 + a \Rightarrow \boxed{a = -1}$

for L: $0 = 1 - 3a + b + c \Rightarrow b + c = -4$

for T: $0 = -2 - b \Rightarrow \boxed{b = -2}$ $\therefore \boxed{c = -2}$

$$\therefore \pi_1 = F \rho^{-1} V^{-2} d^{-2} \Rightarrow \boxed{\pi_1 = \frac{F}{\rho V^2 d^2}}$$ less

DAG-a

$$\pi_2 = \mu\, \rho^a v^b d^c$$

$$\pi_2 = \left[\frac{M}{LT}\right]\left[\frac{M}{L^3}\right]^a\left[\frac{L}{T}\right]^b\left[L\right]^c = M^0 L^0 T^0$$

for M: $1 + a = 0 \Rightarrow \boxed{a = -1}$

for L: $-1 - 3a + b + c = 0 \Rightarrow \boxed{b + c = -2}$

for T: $-1 - b = 0 \Rightarrow \boxed{b = -1}$ $\quad \therefore \boxed{c = -1}$

$$\therefore \pi_2 = \mu \cdot \rho^{-1} v^{-1} d^{-1} \Rightarrow \pi_2 = \frac{\mu}{\rho v d}$$

$$\therefore \boxed{\pi_2 = \frac{1}{Re}}$$

⑦ π – relationships $\quad \pi_1 = f(\pi_2)$

$$\therefore \frac{F}{\rho v^2 d^2} = f\left(\frac{1}{Re}\right)$$

$\frac{1}{2}\rho v^2$

$A = \frac{\pi}{4} d^2$ Area : projected area

$\therefore d^2 = \frac{4A}{\pi}$

$$C_D = \frac{\tau}{\frac{1}{2}\rho v^2}$$

$$\therefore \frac{F}{\rho v^2 d} \propto C_D \qquad \therefore \boxed{C_D = f\left(\frac{1}{Re}\right)}$$

DA6 – b

Ex:. Find the relationship of the pressure drop ($\frac{dP}{dx}$) of a certain fluid flows through a non-smooth, circular cross-sectional area pipe?

non-smooth: ϵ

Sol

$\frac{\Delta P}{\Delta x}$?

$\rho\ u\ d\ \mu\ \epsilon \longrightarrow$

① $\frac{\Delta P}{\Delta x}$, $\boxed{\rho\ \ u\ \ d}$ μ ϵ

② no – temp ∴ M L T

③ write the MLT for each variable

$$\frac{N/m^2}{m} = \frac{N}{m^3} = \frac{kg \cdot \frac{m}{s^2}}{m^3} = \frac{kg}{m^2 \cdot s^2}$$

$\frac{\Delta P}{\Delta x}$	ρ	u	d	μ	ϵ
$\frac{M}{L^2T^2}$	$\frac{M}{L^3}$	$\frac{L}{T}$	L	$\frac{M}{L \cdot T}$	L

④ $n = 6$ $\quad m = 3 \quad n - m = 3$

⑤
$$\pi_1 = \frac{\Delta P}{\Delta x}\, \rho^a\, u^b\, d^c$$
$$\pi_2 = \mu\, \rho^a\, V^b\, d^c$$
$$\pi_3 = \epsilon\, \rho^a\, V^b\, d^c$$

⑥
$$\pi_1 = 1 = M^0 L^0 T^0 = \left[\frac{M}{L^2T^2}\right]\left[\frac{M}{L^3}\right]^a\left[\frac{L}{T}\right]^b\left[L\right]^c$$

M: $0 = 1 + a \Rightarrow \boxed{a = -1}$

L: $0 = -2 - 3a + b + c \Rightarrow \boxed{b + c = -1}$ ∴ $\boxed{c = 1}$

T: $0 = -2 - b \Rightarrow b = -2$

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$$\therefore \pi_1 = \frac{\Delta P}{\Delta x} \cdot \rho^{-1} \cdot V^{-2} d^{1} = \boxed{\frac{\frac{\Delta P}{\Delta x} \cdot d}{\rho V^2}}$$

$$\pi_2 = \mu \, \rho^{a} V^{b} d^{c}$$

$$\pi_2 = M^0 L^0 T^0 = \left[\frac{M}{L.T}\right]\left[\frac{M}{L^3}\right]^{a}\left[\frac{L}{T}\right]^{b}\left[L\right]^{c}$$

$M: \quad 0 = 1 + a \Rightarrow \boxed{a = -1}$

$L: \quad 0 = -1 - 3a + b + c \Rightarrow b + c = -2$

$T: \quad 0 = -1 - b = 0 \Rightarrow \boxed{b = -1} \qquad \therefore \boxed{c = -1}$

$$\therefore \pi_2 = \mu \, \rho^{-1} V^{-1} d^{-1} = \boxed{\frac{\mu}{\rho V d}}$$

$$\pi_3 = \epsilon \, \rho^{a} V^{b} d^{c}$$

$$M^0 L^0 T^0 = [L]\left[\frac{M}{L^3}\right]^{a}\left[\frac{L}{T}\right]^{b}[L]^{c}$$

$M: \quad 0 = a \Rightarrow \boxed{a = 0}$

$L: \quad 0 = 1 - 3a + b + c \Rightarrow b + c = -1$

$T: \quad 0 = -b \Rightarrow \boxed{b = 0} \qquad \therefore \boxed{c = -1}$

$$\therefore \pi_3 = \epsilon \cdot \rho^{0} V^{0} d^{-1} \Rightarrow \pi_3 = \boxed{\frac{\epsilon}{d}}$$

DA8

(7) $\pi_1 = f(\pi_2, \pi_3)$

$$\frac{\frac{\Delta P}{\Delta x} \cdot d}{\rho v^2} = f\left(\frac{\mu}{\rho v d}, \frac{\epsilon}{d}\right)$$

$$= \underbrace{f\left(\frac{1}{Re}, \frac{\epsilon}{d}\right)}_{}$$

What is this?
It is "f" friction factor

$$\therefore \frac{\frac{\Delta P}{\Delta x} \cdot d}{\rho v^2} = f \quad \Rightarrow \quad \frac{\Delta P}{\Delta x} = f \frac{1}{d} \rho v^2$$

$$\boxed{\frac{\Delta P}{\Delta x} = \frac{f}{d} \rho v^2}$$