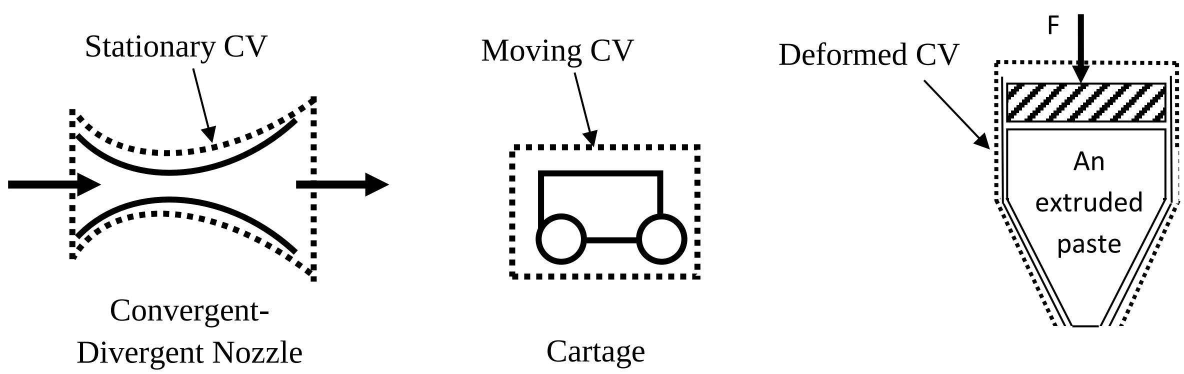
Chapter One

Control Volume and Reynolds Transport Theorem

Control Volume (CV) Concept: It is a concept that describes the whole changes through a volume (certain volume) and to control it to figure out the changes of the whole properties such as flow rate, momentum, energy, etc.

There are three types of control volumes, depend on the nature of the space that undergo the changes as shown in figure below.



Stationary CV

There are three main laws in the mechanics, as we know:

1. The conservation of mass

2. The conservation of momentum (linear and angular)

3. The conservation of energy

When we apply or use these laws in solid, there is a constant volume of the body such as bar, beam, block, etc. However, in fluid, there is no constant volume, so we will create a volume to make the study and the calculations easy. This volume is called the control volume (CV).

Conservation of Mass

Extensive and Intensive Properties

When a property depends on a mass such as the momentum, it is called extensive property and here in this chapter will be noted as "B" or "N", while the mass dependent property will be called as intensive property such as specific volume (m^3/kg) or the momentum and energy per unit mass. These properties can be noted by "b" or "n" such as b=B/m (m: is the mass).

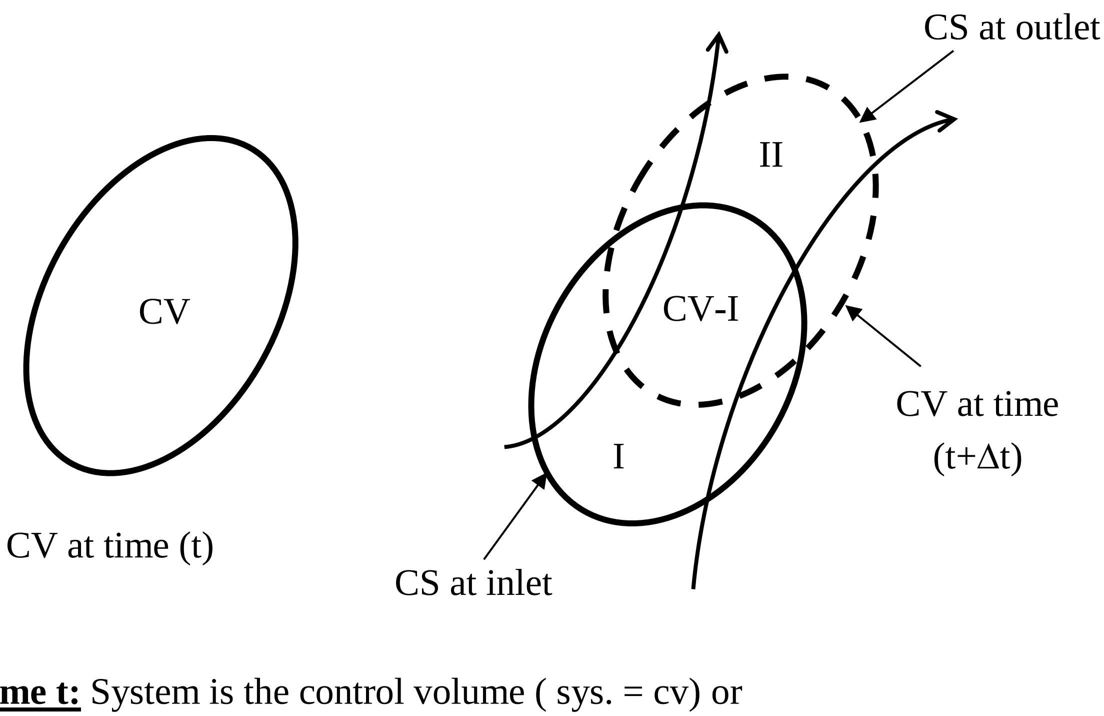
For non-uniform property as B and its intensive property b and per unit mass, we can get:

$b = \frac{dB}{dm}$ then $dB = b \times dm$ -- (1)

$m = \rho V$ then $dm = \rho\, dV$ --- (2)

from (1 and 2) we get:

$dB = b\, \rho\, dV$ then $\boldsymbol{B} = \oint_V \mathbf{b\, \rho\, dV}$



<u>At time t:</u> System is the control volume (sys. = cv) or

B(t) = CV(t) or $B_{sys}=B_{CV}$ at time t

At time (t+Δt): System is the control volume

$$(CV - I) + II = CV - I + II$$

Or

$$B_{sys}(t + \Delta t) = B_{cv}(t + \Delta t) - B_I(t + \Delta t) + B_{II}(t + \Delta t)$$

Now, we take time rate of B change to get the derivative as:

$$B_{sys}(t + \Delta t) - B_{sys}(t) = B_{cv}(t + \Delta t) - B_I(t + \Delta t) + B_{II}(t + \Delta t) - B_{sys}(t)$$

Divide by Δt

$$\frac{B_{sys}(t + \Delta t) - B_{sys}(t)}{\Delta t} = \frac{B_{cv}(t + \Delta t) - B_I(t + \Delta t) + B_{II}(t + \Delta t) - B_{sys}(t)}{\Delta t}$$

The LHS of this equation is the first derivative of B with respect to t when $\Delta t \rightarrow 0$

$$\frac{\Delta B_{sys}}{\Delta t} = \frac{B_{cv}(t + \Delta t) - B_{CV}(t)}{\Delta t} + \frac{B_{II}(t + \Delta t)}{\Delta t} - \frac{B_I(t + \Delta t)}{\Delta t}$$

Rate of B change w.r.t (t) **term (1)**	Change of B w.r.t (t) **term (2)**	B flow rate cross the C.S. at outlet **term (3)**	B flow rate cross the C.S. at inlet **term (4)**

--- The main equation

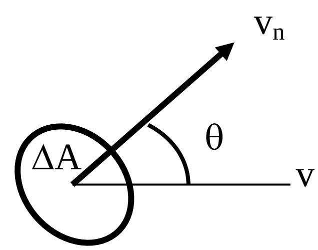
Now, term (1)

$$\lim_{\Delta t \to 0} \frac{\Delta B_{sys}}{\Delta t} = \lim_{\Delta t \to 0} \frac{B_{sys}(t + \Delta t) - B_{sys}(t)}{\Delta t} = \frac{dB_{sys}}{dt}$$

term (2), take limit $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{B_{CV}(t + \Delta t) - B_{CV}(t)}{\Delta t} = \frac{dB_{CV}}{dt} = \frac{d}{dt} \oint_{CV} b\, \rho\, dV$$

term (3) and term (4) are in same analysis, for term (3), outlet the CS:



$v_n = v \cos\theta$ } * Δt [normal component of v to the area ΔA]

$v_n \Delta t = v \cos\theta \; \Delta t$

$\Delta l_n = v \cos\theta \; \Delta t$ } * $\rho \; \Delta A$ [distance of travel of ΔA for time Δt]

Hence: $\rho \, \Delta l_n \, \Delta A = \rho \, v \cos\theta \, \Delta t \, \Delta A$ [volume of flow rate]

But, $\rho \, \Delta l_n \, \Delta A = \rho \, \Delta\forall_n = \Delta m_{out}$ [mass flow rate=density × volume flow rate]

Now, we have: $\Delta m_{out} = \rho \, v \cos\theta \, \Delta t \, \Delta A$ --- (3)

we have $b = B/m$ [any intensive property]

Or $B = b\, m$ → in increment form → $\Delta B_{out} = b \, \Delta \, m_{out}$ -------------------- (4)

Substitute (3) in (4) we get:

$\Delta B_{out} = b \, \rho \, v \cos\theta \, \Delta t \, \Delta A$

For the entire cross-section area at outlet, $\Delta A = dA$ take integration

$$B_{out} = \int b \, \rho \, v \cos\theta \, \Delta t \, dA = B_{II}(t + \Delta t)$$

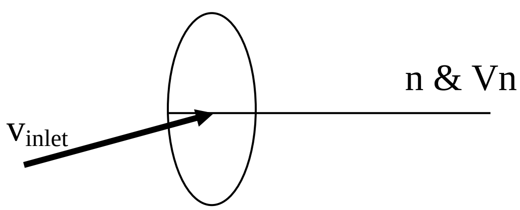
Δt is constant with respect to (A)

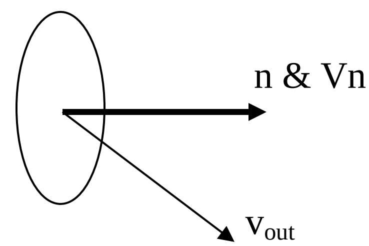
$$\frac{B_{out}}{\Delta t} = \int b\,\rho\,v\cos\theta\,dA$$

Take limit for both sides as $\Delta t \to 0$ Hence: $\boldsymbol{B_{out} = \oint_{CS_{out}} b\,\rho\,v\cos\theta\,dA}$

Same analysis for tem (4) $\boldsymbol{B_{in} = \oint_{CS_{out}} b\,\rho\,v\cos\theta\,dA}$

B_{in} and B_{out} are positive but according to the figure below, B_{in} will be negative ($\cos\theta < 0$) while B_{out} will be positive ($\cos\theta > 0$)





$\theta > 90$ $\theta < 90$

$\cos\theta < 0$ $\cos\theta > 0$

Now the four terms are:

Term (1) : $\frac{dB_{sys}}{dt}$ Term (2) : $\frac{d}{dt}\oint_{CV} b\,\rho\,dV$

Term (3) : $\boldsymbol{\oint_{CS_{out}} b\,\rho\,v\cos\theta\,dA}$ Term (4) : $-\boldsymbol{\oint_{CS_{in}} b\,\rho\,v\cos\theta\,dA}$

$$\hat{v} = v\cos\theta = \bar{v}\cdot\hat{n}$$

Substitute these terms in the main equation, we get:

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\oint_{CV} b\,\rho\,d\forall + \oint_{CS_{out}} b\,\rho\,\bar{v}\,\hat{n}\,dA - \left(-\oint_{CS_{in}} b\,\rho\,\bar{v}\,\hat{n}\,dA\right)$$

$$\frac{dB_{sys}}{dt} = \frac{d}{dt}\oint_{CV} b\,\rho\,d\forall + \oint_{CS_{out}} b\,\rho\,\bar{v}\,\hat{n}\,dA + \oint_{CS_{in}} b\,\rho\,\bar{v}\,\hat{n}\,dA$$

In conclusion,

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{d}{dt}\oint_{CV} b\,\rho\,d\forall + \oint_{CS} b\,\rho\,\bar{\boldsymbol{v}} \bullet \hat{\boldsymbol{n}}\,dA \quad \text{-------------------------(5)}$$

$$\bar{v}\cdot\hat{n} = +\bar{v}\ outlet\ CS \qquad = -\bar{v}\ inlet\ CS$$

Equation (5) is called **Reynolds Transport Theorem (RTT)**.

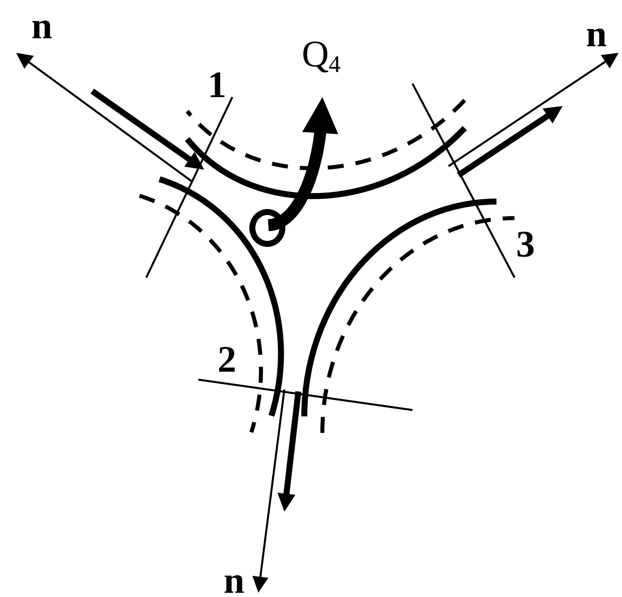
Example 1: Find the velocity at the section 2

$A_1 = 0.2\ m^2$ $v_1 = 5$ m/s

$A_2 = 0.2\ m^2$ $v_2 = ?$

$A_3 = 0.15\ m^2$ $v_3 = 12$ m/s

$Q_4 = 0.1\ m^3/s$



Solution:

Method 1 (the direct method)

$Q_1 = Q_2 + Q_3 + Q_4$

$Q_2 = Q_1 - Q_3 - Q_4$

$v_2 A_2 = v_1 A_1 - v_3 A_3 - Q_4$

$$v_2 = \frac{v_1 A_1 - v_3 A_3 - Q_4}{A_2}$$

$$v_2 = \frac{0.2(5) - 0.15(12) - 0.1}{0.2} = -4.5\ \text{m/s as at outlet (4.5 m/s inlet)}$$

By using RTT

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{d}{dt}\oint_{CV} b\,\rho\, d\forall + \oint_{CS} b\,\rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

b=B/m and since we talk about the mass then B=m and thus b=1

$$\left.\frac{dB}{dt}\right)_{sys} = 0 \qquad \text{conservation of mass}$$

then,

$$0 = \frac{d}{dt}\oint_{CV} 1\,\rho\, d\forall + \oint_{CS} 1\,\rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

$$\frac{d}{dt} = 0 \qquad \text{steady state}$$

$$0 = \oint_{CS} \rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

$$0 = -A_1v_1 + A_2v_2 + A_3v_3 + A_4v_4$$

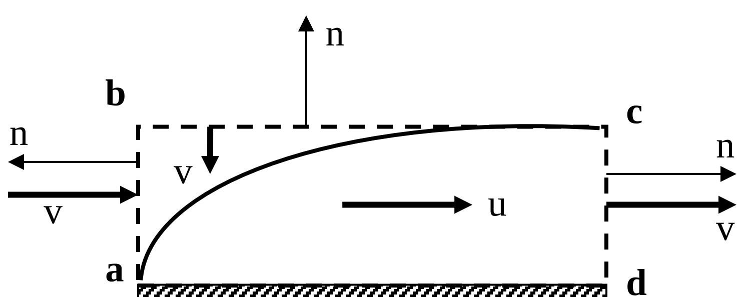
$$0 = -A_1v_1 + A_2v_2 + A_3v_3 + Q_4$$

$$0 = -0.2(5) + 0.2v_2 + 0.15(12) + 0.1$$

$v_2 = -4.5$ m/s outlet or $v_2 = +4.5$ m/s inlet

Example 2: A fluid flow with a velocity 30 m/s on a plate. The velocity profile is $\frac{u}{U} = 2\frac{y}{5} - \left(\frac{y}{5}\right)^2$. The density was 1.24 kg/m3. If the width of the plate is 0.6 m, find the flow rate across the control surfaces? (Assume the control height is 5 mm)

Solution:



Flow rate only

B = m then b = m/m = 1

$$\left.\frac{dB}{dt}\right)_{sys} = \frac{d}{dt}\oint_{CV} b\,\rho\, d\forall + \oint_{CS} b\,\rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

$$0 = 0 + \oint_{CS} b\,\rho\,\overline{\boldsymbol{v}\cdot}\,\hat{\boldsymbol{n}}\,dA$$

$$\oint_{CS} b\,\rho\,\overline{\boldsymbol{v}\cdot}\,\hat{\boldsymbol{n}}\,dA = 0$$

$$\int_{ab} \rho\, v\, \hat{n}\, dA + \int_{bc} \rho\, v\, \hat{n}\, dA + \int_{cd} \rho\, v\, \hat{n}\, dA + \int_{da} \rho\, v\, \hat{n}\, dA = 0$$

v = U | **v=U** | **v=u** | **v=0**

n (-) | **n(-)** | **n(+)** | **no sign**

ρ = constant

$$Q_{ab} = -\oint_{ab} U\,dA = -\int_0^5 U\,w\,dy = -U\,w\,y]_0^5 = -5\,w\,U = 5 \times 0.6 \times 30 = -90\,m^3/s$$

$$Q_{bc} = -\oint_{bc} U\,dA = -\int_0^L U\,w\,dx = -w\,U\,L$$ & L is unknown here

$$Q_{cd} = \int_0^{\delta} U \left[2\frac{y}{5} - \frac{y^2}{25}\right] w\,dy = w\,U \left[\frac{y^2}{5} - \frac{y^3}{75}\right]_0^5 = w\,U\left[\frac{25}{5} - \frac{125}{75}\right]$$

$$= 0.6 \times 30 \times \left[5 - \frac{5}{3}\right] = 0.6 \times 30 \times 5 \times \frac{2}{3} = 60\,m^3/s$$

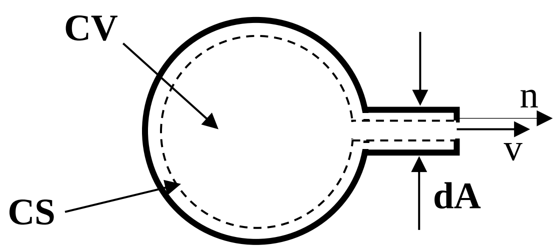
$Q_{da} = 0$ (no flow thru the plate) and

$$Q_{ab} + Q_{bc} + Q_{cd} = 0$$

$$Q_{bc} = -Q_{ab} - Q_{cd} = 90 - 60 = \mathbf{30\,m^3/s}$$

Example 3: A tank has a fluid of ρ = 6 kg/m³ at 800 kpa and 15 °C. At time t=0, the tank valve has opened and the fluid started to escape. Find the density rate of change if the initial volume was 0.05 m³ and the valve area is 65 mm², speed is 300 m/s?

Solution:



$$\left.\frac{dB}{dt}\right)_{sys} = \frac{d}{dt}\oint_{CV} b\,\rho\, d\forall + \oint_{CS} b\,\rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

=0 b=m/m=1

$$0 = \frac{d}{dt}\oint_{CV} 1\,\rho\, d\forall + \oint_{CS} 1\,\rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

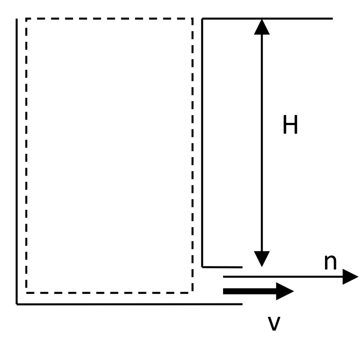
$$0 = \frac{d\rho}{dt}\oint_{CV} d\forall + \oint_{CS} \rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

$0 = \frac{d\rho}{dt}\forall + \rho\, v\, A$ then $\frac{d\rho}{dt} = -\frac{\rho\, v\, A}{\forall}$

$$\therefore \frac{d\rho}{dt} = -\frac{6 \times 300 \times 65 \times 10^{-6}}{0.05} = \boldsymbol{-2.34\ kg/m^3 \cdot s}$$

Example 4: How fast the fluid escape from a tank in form of dh/dt if the hole flow rate is Q and the tank volume is V?

Solution:



$$\left.\frac{dB}{dt}\right)_{sys} = \frac{d}{dt}\oint_{CV} b\,\rho\, d\forall + \oint_{CS} b\,\rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

$\left.\frac{dB}{dt}\right)_{sys} = 0$ b=1

$$0 = \frac{d\rho}{dt}\oint_{CV} d\forall + \oint_{CS} \rho\, \bar{\mathbf{v}}\cdot\hat{\mathbf{n}}\, dA$$

$0 = \frac{d\forall}{dt} + v\, A$ then $\frac{d\forall}{dt} = -v\, A_{hole}$

$\forall = A_{tank} dh$ and A_{tank} is constant →

$$\boldsymbol{\frac{dh}{dt} = -\frac{v\, A_{hole}}{A_{tank}} = -\frac{Q_{hole}}{A_{tank}}}$$