المحاضرة الخامسة

جذور الاعداد المركبة

Roots of complex number

$$\boldsymbol{z^{\frac{1}{n}} = r^{\frac{1}{n}} e^{i\left[\frac{\theta+2k\pi}{n}\right]} = r^{\frac{1}{n}}\left[cos\left(\frac{\theta+2k\pi}{n}\right) + isin\left(\frac{\theta+2k\pi}{n}\right)\right], k = 0, 1, 2, \ldots, n-1}$$

Ex. Find out the roots of $(-1)^{\frac{1}{2}}$

Sol. $z = -1 + 0i$

$$r = 1$$

$$\theta = \tan^{-1}\left(\frac{0}{-1}\right) = \tan^{-1}(0) = \pi$$

$$z = r[cos\theta + isin\theta]$$

$$z = cos\pi + isin\pi$$

$$z^{\frac{1}{2}} = cos\left(\frac{\pi+2k\pi}{n}\right) + isin\left(\frac{\pi+2k\pi}{n}\right) \qquad k = 0,1$$

$$k = 0 \;\; then = cos\left(\frac{\pi+2(0)\pi}{2}\right) + isin\left(\frac{\pi+2(0)\pi}{2}\right)$$

$$z_1 = cos\frac{\pi}{2} + isin\frac{\pi}{2} \Longrightarrow z_1 = 0 + i = i$$

$$k = 1 \;\; thencos\left(\frac{\pi+2(1)\pi}{2}\right) + isin\left(\frac{\pi+2(1)\pi}{2}\right)$$

$$z_2 = cos\frac{3\pi}{2} + isin\frac{3\pi}{2} \Longrightarrow z_2 = 0 - i = -i$$

$$s = \{i, -i\}$$

Ex. Find out the cube roots of $(-8i)$ اكتشف الجذور التكعيبية

Sol. $z = 0 - 8i \quad \Longrightarrow r = \sqrt{0 + 64} = 8$

$$\theta = \tan^{-1}\left(\frac{-8}{0}\right) = -\frac{\pi}{2}$$

$$z = 8\left[cos\frac{-\pi}{2} + isin\frac{-\pi}{2}\right] \Longrightarrow z = 8\left[cos\frac{\pi}{2} - isin\frac{\pi}{2}\right]$$

$$z^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[cos\frac{\pi}{2} - isin\frac{\pi}{2}\right]^{\frac{1}{3}} = 8^{\frac{1}{3}}\left[cos\frac{\frac{\pi}{2} + 2k\pi}{3} - isin\frac{\frac{\pi}{2} + 2k\pi}{3}\right] \quad k = 0,1,2$$

$$k = 0\ then\ z_1 = 2\left[cos\frac{\pi}{6} - isin\frac{\pi}{6}\right] \Longrightarrow z_1 = 2\left[\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] \Longrightarrow z_1 = \sqrt{3} - i$$

$$k = 1\ then\ z_2 = 2\left[cos\frac{\frac{5\pi}{2}}{3} - isin\frac{\frac{5\pi}{2}}{3}\right] \Longrightarrow z_2 = 2\left[cos\frac{5\pi}{6} - isin\frac{5\pi}{6}\right] \Longrightarrow$$

$$= 2\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right] = -\sqrt{3} - i$$

$$k = 2\ then\ z_3 = 2\left[cos\frac{\frac{9\pi}{2}}{3} - isin\frac{\frac{9\pi}{2}}{3}\right] \Longrightarrow z_2 = 2\left[cos\frac{3\pi}{2} - isin\frac{3\pi}{2}\right] \Longrightarrow$$

$$= 2[0 + i] = 2i$$

$$s = \left\{2i, -\sqrt{3} - i, \sqrt{3} - i\right\}$$

Ex. Solve the equation $x^3 + i = 0$

Sol. $x^3 = -i$ this means $x = (-i)^{\frac{1}{3}}$

$$\Rightarrow r = |z| \Rightarrow r = 1 \;\&\; \theta = -\frac{\pi}{2} \Rightarrow \therefore\; x^3 = cos\left(-\frac{\pi}{2}\right) + isin\left(-\frac{\pi}{2}\right)$$

$$x^3 = cos\frac{\pi}{2} - isin\frac{\pi}{2} \qquad \text{اوجد الجذور التكعيبية} \quad k = 0,1.2$$

$$k = 0 \;\Rightarrow x_1 = cos\frac{\frac{\pi}{2} + 2(0)\pi}{3} - isin\frac{\frac{\pi}{2} + 2(0)\pi}{3} = cos\frac{\pi}{6} - isin\frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$k = 1 \;\Rightarrow x_2 = cos\frac{\frac{\pi}{2} + 2\pi}{3} - isin\frac{\frac{\pi}{2} + 2\pi}{3} = cos\frac{5\pi}{6} - isin\frac{5\pi}{6} = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

$$k = 2 \;\Rightarrow x_3 = cos\frac{\frac{\pi}{2} + 4\pi}{3} - isin\frac{\frac{\pi}{2} + 4\pi}{3} = cos\frac{9\pi}{6} - isin\frac{9\pi}{6} = cos\frac{3\pi}{2} - isin\frac{3\pi}{2}$$

$$= 0 + i = i$$

$$s = \left\{i, -\frac{\sqrt{3}}{2} - \frac{1}{2}i, \frac{\sqrt{3}}{2} - \frac{1}{2}i\right\}$$

Ex. Find out the cube roots of $\left(\sqrt{3} + i\right)^{-\frac{3}{2}}$

Sol. $z = \sqrt{3} + i$,$\Rightarrow r = 2$, $\Rightarrow \theta = \frac{\pi}{6}$

$z = 2\left[cos\frac{\pi}{6} + isin\frac{\pi}{6}\right]$.

Now by Demoivre's formula

$$Z^{-3} = 2^{-3}\left[cos\frac{\pi}{6} + isin\frac{\pi}{6}\right]^{-3}$$

$$Z^{-3} = 2^{-3}\left[cos(-3)\frac{\pi}{6} + isin(-3)\frac{\pi}{6}\right]$$

$$Z^{-3} = \frac{1}{8}\left[cos\frac{\pi}{2} - isin\frac{\pi}{2}\right]$$

hence

$$Z^{-\frac{3}{2}} = \left(\frac{1}{8}\right)^{\frac{1}{2}}\left[cos\left(\frac{\frac{\pi}{2}+2K\pi}{2}\right) - isin\left(\frac{\frac{\pi}{2}+2K\pi}{2}\right)\right].$$ Where K=0, 1

$$K = 0\,, Z_1 = \left(\frac{1}{2\sqrt{2}}\right)\left[cos\left(\frac{\pi}{4}\right) - isin\left(\frac{\pi}{4}\right)\right] \Rightarrow \left(\frac{1}{2\sqrt{2}}\right)\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right]$$

$$\frac{1}{4} - \frac{1}{4}i$$

$$K = 1, Z_2 = \left(\frac{1}{2\sqrt{2}}\right)\left[cos\left(\frac{5\pi}{4}\right) - isin\left(\frac{5\pi}{4}\right)\right] \Rightarrow \left(\frac{1}{2\sqrt{2}}\right)\left[\frac{-1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right]$$

$$\frac{-1}{4} + \frac{1}{4}i$$

Ex. Find out the cube roots of $(-i)^{-\frac{2}{3}}$

Sol. $z = 0 - i \;\; , \Rightarrow r = 1\,, \;\; \Rightarrow \theta = -\frac{\pi}{2}$

$z = 1\left[cos\frac{\pi}{2} - isin\frac{\pi}{2}\right].$

By Demoivre's formula

$$Z^{-2} = 1^{-2}\left[cos\frac{\pi}{2} - isin\frac{\pi}{2}\right]^{-2}$$

$$= [cos\pi + isin\pi]$$

$Z^{-\frac{2}{3}} = (1)^3 \left[cos\left(\frac{\pi+2K\pi}{3}\right) + isin\left(\frac{\pi+2K\pi}{3}\right)\right]$. Where K=0, 1, 2

$$K = 0\ , Z_1 = \left[cos\left(\frac{\pi}{3}\right) + isin\left(\frac{\pi}{3}\right)\right] \Longrightarrow \left[\frac{1}{2} + \frac{\sqrt{3}}{2}i\right]$$

$$K = 1, Z_2 = [cos(\pi) + isin(\pi)] \Longrightarrow [0 - i] = -i$$

$$K = 2, Z_3 = \left[cos\left(\frac{5\pi}{3}\right) + isin\left(\frac{5\pi}{3}\right)\right] \Longrightarrow \left[\frac{1}{2} - \frac{\sqrt{3}}{2}i\right] =$$