Solution of Linear System by Matrices:

Cramer's Rule

THEOREM
Cramer's Rule

If a system of n linear equations in n variables has a coefficient matrix with a nonzero determinant $|A|$, then the solution of the system is given by

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)},$$

where the ith column of A_i is the column of constants in the system of equations.

EXAMPLE

Use Cramer's Rule to solve the system of linear equations for x.

$$\begin{aligned} -x + 2y - 3z &= 1 \\ 2x + z &= 0 \\ 3x - 4y + 4z &= 2 \end{aligned}$$

SOLUTION

$$Ax = b \quad , \quad A = \begin{bmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{bmatrix} , x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The determinant of the coefficient matrix is

$$|A| = \begin{vmatrix} -1 & 2 & -3 \\ 2 & 0 & 1 \\ 3 & -4 & 4 \end{vmatrix} = 10.$$

Because $|A| \neq 0$, you know the solution is unique, and Cramer's Rule can be applied to solve for x, as follows.

$$x = \frac{|A_1|}{|A|} \quad , \quad y = \frac{|A_2|}{|A|} \quad , \quad z = \frac{|A_3|}{|A|}$$

$$A_1 = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{bmatrix}, A_2 = \begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, A_3 = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{bmatrix}$$

$$x = \frac{\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & 1 \\ 2 & -4 & 4 \end{vmatrix}}{10} = \frac{(1)(-1)^5\begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix}}{10} = \frac{(1)(-1)(-8)}{10} = \frac{4}{5}$$

$$y = \frac{\begin{vmatrix} -1 & 1 & -3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{vmatrix}}{10} = \frac{-15}{10} = \frac{-3}{2}$$

$$z = \frac{\begin{vmatrix} -1 & 2 & 1 \\ 2 & 0 & 0 \\ 3 & -4 & 2 \end{vmatrix}}{10} = \frac{-16}{10} = \frac{-8}{5}$$

EXAMPLE

Use Cramer's Rule to solve the system of linear equations for x.

$$\begin{aligned} 2x + y &= 7 \\ 3x - 4y &= 5 \end{aligned}$$

Calculating $\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11$. Since $\Delta \neq 0$ we can proceed with Cramer's solution.

$$\Delta = \begin{vmatrix} 2 & 1 \\ 3 & -4 \end{vmatrix} = -11 \qquad x = \frac{1}{\Delta}\begin{vmatrix} 7 & 1 \\ 5 & -4 \end{vmatrix}, \quad y = \frac{1}{\Delta}\begin{vmatrix} 2 & 7 \\ 3 & 5 \end{vmatrix}$$

i.e. $x = \dfrac{(-28-5)}{(-11)}$, $y = \dfrac{(10-21)}{(-11)}$ implying: $x = \dfrac{-33}{-11} = 3$, $y = \dfrac{-11}{-11} = 1.$

Exercises

Use Cramer's rule to solve the system

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 3 \\ 2x_1 + x_2 - x_3 &= 5 \\ 3x_1 - x_2 + 2x_3 &= 12. \end{aligned}$$

Gaussian Elimination method

We can use Gauss Elimination Method to solve the system of linear equation as see in the following example

Example : Solve the following systems of linear equations by Gaussian elimination method:

$$2x - 2y + 3z = 2$$
$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$

Solution:

Consider the matrix equation (augmented matrix): (R= row)

$$\begin{bmatrix} 2 & -2 & 3 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \begin{array}{c} \underrightarrow{R1} \\ \underrightarrow{R2} \\ \underrightarrow{R3} \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 3 & 2 \\ 1 & 2 & -1 & 3 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R1 \quad swap \quad R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & -2 & 3 & 2 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R2 \mapsto R2-2R1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 3 & -1 & 2 & 1 \end{array}\right]$$

$$\underset{=0}{\underline{[2-2(1)]}} \quad \underset{=-6}{\underline{[-2-2(2)]}} \quad \underset{=5}{\underline{[+3-2(-1)]}} \quad \underset{=-4}{\underline{[+2-2(3)]}}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 3 & -1 & 2 & 1 \end{array}\right] \xrightarrow{R3 \mapsto R3-3R1} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right]$$

$$\underset{=0}{\underline{[3-3(1)]}} \quad \underset{=-7}{\underline{[-1-3(2)]}} \quad \underset{=5}{\underline{[+2-3(-1)]}} \quad \underset{=-8}{\underline{[+1-3(3)]}}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & -7 & 5 & -8 \end{array}\right] \xrightarrow{R3 \mapsto 6R3-7R2} \left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 0 & -6 & 5 & -4 \\ 0 & 0 & -5 & -20 \end{array}\right]$$

$$\underset{=0}{\underline{[0-0]}} \quad \underset{=0}{\underline{[6(-7)-7(-6)]}} \quad \underset{=-5}{\underline{[6(5)-7(5)]}} \quad \underset{=-20}{\underline{[6(-8)-7(-4)]}}$$

The equivalent system is written by using the echelon form:

x+2y-z=3 ....(1)
-6y+5z=-4 ....(2)
-5z=-20 ....(3)

From eq.3: $[-5z = -20 \Longrightarrow \mathbf{z=4}]$

By applying the value of z in (2), we can get:
$-6y + 5(4) = -4 \Longrightarrow -6y + 20 = -4 \Longrightarrow \mathbf{y=4}$
By applying the value of y and z in (1), we can get:
$x + 2(4) - (4) = 3 \Longrightarrow x + 4 = 3 \Longrightarrow \mathbf{x=-1}$
Hence the solution is (-1, 4, 4)

Example : Solve the following systems of linear equations by Gaussian elimination method:

$$x_1 - 2x_2 + x_3 = 0$$
$$2x_1 + x_2 - 3x_3 = 5$$
$$4x_1 - 7x_2 + x_3 = -1$$

Solution:
The augmented matrix which represents this system is:

$$\begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & -3 \\ 4 & -7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ -1 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right] \begin{array}{c} \underrightarrow{R1} \\ \underrightarrow{R2} \\ \underrightarrow{R3} \end{array}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 1 & -3 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right] \xrightarrow{-2R1+R2} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right]$$

$$\underline{[-2(1)+2)]} \quad \underline{[-2(-2)+1]} \quad \underline{[-2(1)+(-3)]} \quad \underline{[-2(0)+5]}$$
$$= 0 \qquad = 5 \qquad = -5 \qquad = 5$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 4 & -7 & 1 & -1 \end{array}\right] \xrightarrow{-4R1+R3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{array}\right]$$

$$\underset{=0}{\underline{[-4(1)+(4))]}} \quad \underset{=1}{\underline{[-4(-2)+(-7)]}} \quad \underset{=-3}{\underline{[-4(1)+(1)]}} \quad \underset{=-1}{\underline{[-4(0)+(-1)]}}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 1 & -3 & -1 \end{array}\right] \xrightarrow{R2 \quad swap \quad R3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -5 & 5 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -5 & 5 \end{array}\right] \xrightarrow{-5R2+R3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 10 & 10 \end{array}\right]$$

$$\underset{=0}{\underline{[-5(0)+(0))]}} \; tiny \quad \underset{=0}{\underline{[-5(1)+(5)]}} \quad \underset{=10}{\underline{[-5(-3)+(-5)]}} \quad \underset{=10}{\underline{[-5(-1)+(5)]}}$$

The equivalent system is written by using the echelon form:

$$x_1 - 2x_2 + x_3 = 0 \quad ....(1)$$
$$x_2 - 3x_3 = -1 \quad ......(2)$$
$$10x_3 = 10 \quad ....(3)$$

From eq.3: $10x_3 = 10 \Longrightarrow x_3 = 1$

By applying the value of x_3 in (2), we can get:
$x_2 - 3x_3 = -1 \Longrightarrow [x_2 - 3(1) = -1] \Longrightarrow x_2 = 2$

By applying the value of x_2 and x_3 in (1), we can get:
$x_1 - 2(2) + 1 = 0 \Longrightarrow [x_1 - 3 = 0] \Longrightarrow x_1 = 3$

The solution of this system is therefore $(x_1, x_2, x_3) = (3, 2, 1)$