Solution of Wave Equation

In this lecture we discuss the one-dimensional wave equation $w_{tt} = k^2 u_{xx}$, where k is a constant non-negative real coefficient representing the propagation speed of the wave. The wave equation in one spatial dimension can be derived for a variety of different physical situations. Most famously, it can be derived for a vibrating string when each of its elements is pulled in opposite directions by a tension force.

We will find the general solution to the wave equation in two methods.

1. Method of separation of variables

The method of separation of variables relies upon the assumption that a function of the form: $w(x,t) = F(x)G(t)$.

Example 1: Apply the method of separation of variables to solve the wave equation $w_{tt} = k^2 u_{xx}$, with boundary conditions $w(0,t) = w(1,t) = 0$ and $w_t(x,0) = 0$.

Solution: Assume the solution is $w(x,t) = F(x)G(t)$

$$w_{xx} = F''G \text{ and } w_{tt} = FG''$$

$$\frac{F''}{F} = \frac{G''}{k^2 G} = -\lambda^2$$

$$F'' + \lambda^2 F = 0 \quad \text{and} \quad G'' + \lambda^2 k^2 G = 0$$

$$F = A\sin\lambda x + B\cos\lambda x \quad \text{and} \quad G = C\sin(\lambda k t) + D\cos(\lambda k t)$$

$$w(0,t) = w(1,t) = 0 \ \Rightarrow\ F(0) = 0 \text{ and } F(1) = 0$$

$$F(0) = 0 \ \Rightarrow\ B = 0 \ \Rightarrow\ F = A\sin\lambda x$$

$$F(1) = 0 \ \Rightarrow\ A\sin\lambda x = 0 \ \Rightarrow\ \lambda = n\pi$$

$$F = A\sin(n\pi x)$$

$$G = C\sin(n\pi k t) + D\cos(n\pi k t)$$

$$w_t(x,0) = 0 \ \Rightarrow\ G'(0) = 0$$

$$G' = Cn\pi k\cos(n\pi k t) - Dn\pi k\sin(n\pi k t)$$

$$G'(0) = 0 \ \Rightarrow C = 0 \ \Rightarrow\ G = D\cos(n\pi k t)$$

$$w(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\cos(n\pi k t)$$

2. D'Alembert's Solution

Suppose we have the wave equation $w_{tt} = c^2 u_{xx}$, and we wish to solve it given the conditions $w(x,0) = F(x)$ and $w_t(x,0) = G(x)$.

We change variables to $r = x + ct$ and $s = x - ct$.

The general solution of this PDE is $u(x,t) = F(x + ct) + G(x - ct)$

Example 2: Solve IVP $w_{tt} - 4w_{xx} = 0$ with $w(x,0) = 0$ and $w_t(x,0) = \tan x$.

Solution: $w(x,t) = F(x + 2t) + G(x - 2t)$

$$w(x,0) = 0 \Rightarrow 0 = F(x + 0) + G(x - 0)$$

$$F(x) + G(x) = 0 \qquad \cdots (1)$$

$$w_t(x,t) = 2F'(x + 2t) - 2G'(x - 2t)$$

$$w_t(x,0) = \tan x \Rightarrow 2F'(x) - 2G'(x) = \tan x$$

$$2F(x) - 2G(x) = -\ln\cos(x) \qquad \cdots (2)$$

$$2 \times \text{equ}(1) + \text{equ}(2) \Rightarrow 4F(x) = -\ln\cos(x)$$

$$F(x) = -\frac{1}{4}\ln\cos(x) \quad \text{and} \quad G(x) = \frac{1}{4}\ln\cos(x)$$

$$\text{So, } F(x + 2t) = -\frac{1}{4}\ln\cos(x + 2t) \quad \text{and} \quad G(x - 2t) = \frac{1}{4}\ln\cos(x - 2t)$$

Then $$w(x,t) = \frac{1}{4}\ln\cos(x - 2t) - \frac{1}{4}\ln\cos(x + 2t)$$

Or $$w(x,t) = \frac{1}{4}\ln\frac{\cos(x - 2t)}{\cos(x + 2t)}$$

Example 3: Solve IVP $w_{tt} - a^2 u_{xx} = 0$ with $w(x,0) = 0$ and $w_t(x,0) = \dfrac{1}{1+x^2}$

Solution: $w(x,t) = F(x+at) + G(x-at)$

$w(x,0) = 0 \quad ⇨ \; 0 = F(x+0) + G(x-0)$

$F(x) + G(x) = 0 \qquad \cdots (1)$

$w_t(x,t) = a\,F'(x+at) - aG'(x-at)$

$w_t(x,0) = \dfrac{1}{1+x^2} \quad ⇨ \quad aF'(x) - aG'(x) = \dfrac{1}{1+x^2}$

$aF(x) - aG(x) = \tan^{-1} x \qquad \cdots (2)$

$a \times \text{equ}(1) + \text{equ}(2) \quad ⇨ \quad 2a\,F(x) = \tan^{-1} x$

$F(x) = \dfrac{1}{2a}\tan^{-1} x$ and $G(x) = -\dfrac{1}{2a}\tan^{-1} x$

So, $F(x+at) = \dfrac{1}{2a}\tan^{-1}(x+at)$ and $G(x-at) = \dfrac{-1}{2a}\tan^{-1}(x-at)$

Then $w(x,t) = \dfrac{1}{2a}\left(\tan^{-1}(x+at) - \tan^{-1}(x-at)\right)$

Example 4: Using D'Alembert's solution to solve the wave equation $w_{tt} = 4w_{xx}$

with $w(x,0) = w_t(x,0) = \sin x$

Solution: $w(x,t) = F(x+2t) + G(x-2t)$

$w(x,0) = \sin x \quad ⇨ \quad \sin x = F(x+0) + G(x-0)$

$F(x) + G(x) = \sin x \qquad \cdots (1)$

$w_t(x,t) = 2\,F'(x+2t) - 2G'(x-2t)$

$w_t(x,0) = \sin x \quad ⇨ \quad 2\,F'(x) - 2G'(x) = \sin x$

$2\,F(x) - 2G(x) = -\cos(x) \qquad \cdots (2)$

$2 \times \text{equ}(1) + \text{equ}(2) \quad ⇨ \quad 4\,F(x) = 2\sin x - \cos x$

$F(x) = (1/2)\sin x - (1/4)\cos x$ and $G(x) = (1/2)\sin x + (1/4)\cos x$

$F(x+2t) = (1/2)\sin(x+2t) - (1/4)\cos(x+2t)$

$G(x-2t) = (1/2)\sin(x-2t) + (1/4)\cos(x-2t)$

$w(x,t) = (1/2)[\sin(x+2t) + \sin(x-2t)] + (1/4)[\cos(x-2t) - \cos(x+2t)]$

Example 5: Using D'Alembert's solution to solve the wave equation $w_{tt} = 2w_{xx}$

with $w(x,0) = \cos 2x$ and $u_t(x,0) = -4\sqrt{2}\sin 2x$

Solution: $w(x,t) = F(x+\sqrt{2}\,t) + G(x-\sqrt{2}\,t)$

$w(x,0) = \cos 2x \quad ⇨ \cos 2x = F(x+0) + G(x-0)$

$F(x) + G(x) = \cos 2x \qquad \cdots (1)$

$w_t(x,t) = \sqrt{2}\,F'(x+\sqrt{2}\,t) - \sqrt{2}\,G'(x-\sqrt{2}\,t)$

$w_t(x,0) = -4\sqrt{2}\sin 2x ⇨ \sqrt{2}\,F'(x) - \sqrt{2}\,G'(x) = -4\sqrt{2}\sin 2x$

$\sqrt{2}\,F(x) - \sqrt{2}\,G(x) = 2\sqrt{2}\cos 2x$

$F(x) - G(x) = 2\cos 2x \qquad \cdots (2)$

$\text{equ}(1) + \text{equ}(2) ⇨ \quad 2F(x) = 3\cos 2x$

$$F(x) = \frac{3}{2}\cos 2x \quad \text{and} \quad G(x) = -\frac{1}{2}\cos 2x$$

$$F(x+\sqrt{2}\,t) = \frac{3}{2}\cos 2(x+\sqrt{2}\,t)$$

$$G(x-\sqrt{2}\,t) = -\frac{1}{2}\cos 2(x-\sqrt{2}\,t)$$

$$w(x,t) = \frac{3}{2}\cos 2(x+\sqrt{2}\,t) - \frac{1}{2}\cos 2(x-\sqrt{2}\,t)$$

H.W: Solve IVPs by using D'Alembert's solution

1. $w_{tt} - a^2 u_{xx} = 0$ with $w(x,0) = 0$ and $w_t(x,0) = \sin x$
2. $w_{tt} - 9w_{xx} = 0$ with $w(x,0) = 0$ and $w_t(x,0) = e^{2x}$
3. $w_{tt} - 3w_{xx} = 0$ with $w(x,0) = e^{2x}$ and $w_t(x,0) = \sqrt{3}\,e^{2x}$