Discrete Structures
1st Stage

Lec_ 2

1.2 Propositional Equivalences

An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value. Because of this, methods that produce propositions with the same truth value as a given compound proposition are used extensively in the construction of mathematical arguments. Note that we will use the term "compound proposition" to refer to an expression formed from propositional variables using logical operators, such as $p \wedge q$. We begin our discussion with a classification of compound propositions according to their possible truth values.

Definition 1.12. *A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a* ***tautology****. A compound proposition that is always false is called a* ***contradiction****. A compound proposition that is neither a tautology nor a contradiction is called a* ***contingency****.*

Tautologies and contradictions are often important in mathematical reasoning. An example of a tautology is $p \vee \neg p$ whereas an example for a contradiction is $p \wedge \neg p$. The following truth table illustrates this.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Table 1.9: Examples of Tautology and Contradiction

1.2.1 Logical Equivalences

Compound propositions that have the same truth values in all possible cases are called logically equivalent. We can also define the notion as follows.

Definition 1.13. *The compound propositions p and q are called* ***logically equivalent*** *if $p \leftrightarrow q$ is a tautology. The notation $p \equiv q$ denotes that p and q are logically equivalent.*

One way to determine whether two compound propositions are equivalent is to use a truth table.

Example 1.4. Show that $\neg(p \vee q)$ and $\neg p \wedge \neg q$ are logically equivalent.
Solution: The truth tables for these compound propositions are displayed in Table 1.10. Because the truth values of the compound propositions $\neg(p \vee q)$ and $\neg p \wedge \neg q$ agree for all possible combinations of the truth values of p and q, it follows that $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$ is a tautology and that these compound propositions are logically equivalent.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$	$\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$
T	T	T	F	F	F	F	T
T	F	T	F	F	T	F	T
F	T	T	F	T	F	F	T
F	F	F	T	T	T	T	T

Table 1.10: The truth table

Example 1.5. Show that $p \to q$ and $\neg p \vee q$ are logically equivalent.
Solution: We construct the truth table for these compound propositions in Table 1.11. Because the truth values of $\neg p \vee q$ and $p \to q$ agree, they are logically equivalent.

p	q	$\neg p$	$\neg p \vee q$	$p \to q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Table 1.11: The truth table

We will now establish a logical equivalence of two compound propositions involving three different propositional variables p, q, and r. To use a truth table to establish such a logical equivalence, we need eight rows, one for each possible combination of truth values of these three variables. In general, 2^n rows are required in the truth table to establish logical equivalence involving n propositional variables.

Table 1.12 demonstrates that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent.

Example 1.6. Show that $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ are logically equivalent. This is the distributive law of disjunction over conjunction.
Solution: We construct the truth table for these compound propositions in Table

$$\boxed{p \vee (q \wedge r) \text{ and } (p \vee q) \wedge (p \vee r)}$$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Table 1.12: The truth table

1.12. Because the truth values of $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ agree, these compound propositions are logically equivalent.

Equivalence	Name	Equivalence	Name
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws	$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws	$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$\neg(\neg p) \equiv p$	Double negation law	$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws	$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Table 1.13: Logical Equivalences

Table 1.13 contains some important equivalences. In these equivalences, $\mathbf{T}$ denotes the compound proposition that is always true and $\mathbf{F}$ denotes the compound proposition that is always false. Note that $p_1 \vee p_2 \vee \ldots \vee p_n$ and $p_1 \wedge p_2 \wedge \ldots \wedge p_n$ are well defined whenever $p_1, p_2, \ldots, p_n$ are propositions. Also De Morgan's laws extend to

$$\neg(p_1 \vee p_2 \vee \ldots \vee p_n) \equiv (\neg p_1 \wedge \neg p_2 \wedge \ldots \wedge \neg p_n)$$

and

$$\neg(p_1 \wedge p_2 \wedge \ldots \wedge p_n) \equiv (\neg p_1 \vee \neg p_2 \vee \ldots \vee \neg p_n).$$

Example 1.7. Show that $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent by developing a series of logical equivalences.

Solution: We will use one of the equivalences in Table 1.13 at a time, starting with $\neg(p \vee (\neg p \wedge q))$ and ending with $\neg p \wedge \neg q$. We have the following equivalences.

Equivalence	
$p \to q \equiv \neg p \vee q$ $p \to q \equiv \neg q \to \neg p$ $p \vee q \equiv \neg p \to q$ $p \wedge q \equiv \neg(p \to \neg q)$ $\neg(p \to q) \equiv p \wedge \neg q$ $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$ $(p \to r) \wedge (q \to r) \equiv (p \vee q) \to r$ $(p \to q) \vee (p \to r) \equiv p \to (q \vee r)$ $(p \to r) \vee (q \to r) \equiv (p \wedge q) \to r$	Logical Equivalences Involving Conditional Statements
$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$ $p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$ $p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$ $\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$	Logical Equivalences Involving Biconditional Statements

Table 1.14: Logical Equivalences

$$
\begin{array}{rcll}
\neg(p \vee (\neg p \wedge q)) & \equiv & \neg p \wedge \neg(\neg p \wedge q) & \text{by the second De Morgan law} \\
& \equiv & \neg p \wedge [\neg(\neg p) \vee \neg q] & \text{by the first De Morgan law} \\
& \equiv & \neg p \wedge (p \vee \neg q) & \text{by the double negation law} \\
& \equiv & (\neg p \wedge p) \vee (\neg p \wedge \neg q) & \text{by the second distributive law} \\
& \equiv & \mathbf{F} \vee (\neg p \wedge \neg q) & \text{by the second negation law} \\
& \equiv & (\neg p \wedge \neg q) \vee \mathbf{F} & \text{by the commutative law} \\
& & & \text{for disjunction} \\
& \equiv & \neg p \wedge \neg q & \text{by the identity law for } \mathbf{F}
\end{array}
$$

Consequently $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$ are logically equivalent.

Logical equivalences involving conditional statements and biconditional statements are given in the table 1.14. These equivalences are important as they form basic tools for proving theorems. Few theorems involve "if and only if" $p \leftrightarrow q$. To prove the theorem of this type we use the equivalence $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$. So it is enough to prove the statements $p \to q$ and $q \to p$ separately.

Remark 1.1. *A logical equivalence can be proved by using either a truth table or by using a chain of known logical equivalences. Also a tautology can be proved by using either a truth table or by using logical equivalences.*

Example 1.8. Show that $(p \wedge q) \to (p \vee q)$ is a tautology.

Solution: To show that this statement is a tautology, we will use logical equivalences to demonstrate that it is logically equivalent to **T**.

$$
\begin{array}{rcll}
(p \wedge q) \rightarrow (p \vee q) & \equiv & \neg(p \wedge q) \vee (p \vee q) & \text{since } p \rightarrow q \equiv \neg p \vee q \\
& \equiv & (\neg p \vee \neg q) \vee (p \vee q) & \text{by the first De Morgan law} \\
& \equiv & (\neg p \vee p) \vee (\neg q \vee q) & \text{by the associative and comm-} \\
& & & \text{utative laws for disjunction} \\
& \equiv & \mathbf{T} \vee \mathbf{T} & \text{by the commutative and} \\
& & & \text{negation laws for disjunction.} \\
& \equiv & \mathbf{T} & \text{by the domination law}
\end{array}
$$

Thus we have shown that $(p \wedge q) \rightarrow (p \vee q)$ is a tautology.
(Note: This could also be done using a truth table.)

Logic has practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study. In the next chapter we will introduce the concepts which will help us to express the meaning of statements in mathematics and natural language.