Applications of Taylor - Maclaurin Polynomials

Maclaurin Series for Common Functions

Function	Maclaurin Series	Interval of Convergence
$f(x) = \dfrac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$	$-1 < x < 1$
$f(x) = e^x$	$\sum_{n=0}^{\infty} \dfrac{x^n}{n!}$	$-\infty < x < \infty$
$f(x) = \sin x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$	$-\infty < x < \infty$
$f(x) = \cos x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n}}{(2n)!}$	$-\infty < x < \infty$
$f(x) = \ln(1+x)$	$\sum_{n=0}^{\infty} (-1)^{n+1} \dfrac{x^n}{n}$	$-1 < x < 1$
$f(x) = \tan^{-1} x$	$\sum_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$	$-1 < x < 1$

1. Solving of Algebraic Equations

Sometimes we have an equation that contains trigonometric, exponential, or logarithmic functions. There is no specific way to find the values that satisfy that equation other than converting those functions to a polynomial, and the following example illustrates that.

Example 1: Find the approximate value of $x > 0$, if $x + 40e^{-x/40} = 40.25$

Solution

$$e^x = \frac{1}{0!} + \frac{1}{1!}x + \frac{1}{2!}x^2$$

$$e^{-x/40} \cong 1 - \frac{x}{40} + \frac{(-x/40)^2}{2} = 1 - \frac{x}{40} + \frac{x^2}{3200}$$

$$x + 40e^{-x/40} = 40.25$$

$$x + 40\left(1 - \frac{x}{40} + \frac{x^2}{3200}\right) \cong 40.25$$

$$x + 40 - x + \frac{x^2}{80} \cong 40.25$$

$$\frac{x^2}{80} \cong 0.25$$

$$x^2 \cong 20$$

$$x \cong 4.47$$

2. Solving of Differential Equations with Power Series

Power series are an extremely useful tool for solving many types of differential equations. In this technique, we look for a solution of the form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and determine what the coefficients would need to be.

Example 2: Use power series to solve the initial-value problem

$$y' = y, \qquad y(0) = 3$$

Solution

Suppose that there exists a power series solution

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots$$

$$y' = y \;\Rightarrow\; c_1 + 2c_2 x + 3c_3 x^2 + 4c_3 x^3 + \cdots = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \cdots$$

Coefficients of x^0: $c_1 = c_0 \;\Rightarrow\; c_1 = \frac{1}{1!} c_0$

Coefficients of x^1: $2c_2 = c_1 \;\Rightarrow\; c_2 = \frac{1}{2} c_1 = \frac{1}{2} c_0 = \frac{1}{2!} c_0$

Coefficients of x^2: $3c_3 = c_2 \;\Rightarrow\; c_3 = \frac{1}{3} c_2 = \frac{1}{2 \times 3} c_0 = \frac{1}{3!} c_0$

Coefficients of x^3: $4c_4 = c_3 \;\Rightarrow\; c_4 = \frac{1}{4} c_3 = \frac{1}{2 \times 3 \times 4} c_0 = \frac{1}{4!} c_0$

So, $\quad y = c_0 + \frac{1}{1!} c_0 x + \frac{1}{2!} c_0 x^2 + \frac{1}{3!} c_0 x^3 + \frac{1}{4!} c_0 x^4 + \cdots$

$$y = c_0 \left(\frac{1}{0!} + \frac{1}{1!} x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots \right)$$

$y = c_0 e^x$

$y(0) = 3 \quad \Rightarrow 3 = c_0 e^0 \quad \Rightarrow \; c_0 = 3$

Therefore, the solution is $y = 3e^x$

3. Evaluating Integrals

We show how power series can be used to evaluate integrals involving functions whose antiderivatives cannot be expressed using elementary functions.

Example 3: Evaluate $\int_0^1 e^{-x^2}\,dx$ to within an error of 0.01.

Solution

The Maclaurin series for e^x is given by:

$$e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

So, the Maclaurin series for e^{-x^2} is given by:

$$e^{-x^2} = \frac{1}{0!} + \frac{-x^2}{1!} + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \frac{(-x^2)^4}{4!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + \cdots$$

$$\int_0^1 e^{-x^2}\,dx = \int_0^1 \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} + \cdots\right) dx$$

$$= x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \frac{x^9}{216} + \cdots \Big|_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} + \cdots$$

The sum of the first four terms is approximately 0.74.

By the alternating series test, this estimate is accurate to within an error of less than

$$\frac{1}{216} \cong 0.0046296 < 0.01.$$

Example 4: Evaluate $\int_0^1 \cos\sqrt{x}\,dx$ to within an error of 0.001.

Solution

The Maclaurin series for $\cos x$ is given by:

$$\cos x = \frac{1}{0!} - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

So, the Maclaurin series for $\cos\sqrt{x}$ is given by:

$$\cos\sqrt{x} = \frac{1}{0!} - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \cdots = 1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots$$

$$\int_0^1 \cos\sqrt{x}\,dx = \int_0^1 \left(1 - \frac{x}{2} + \frac{x^2}{24} - \frac{x^3}{720} + \cdots\right) dx$$

$$= x - \frac{x^2}{4} + \frac{x^3}{72} - \frac{x^4}{2880} + \cdots \Big|_0^1$$

$$= 1 - \frac{1}{4} + \frac{1}{72} - \frac{1}{2880} + \cdots$$

The sum of the first three terms is approximately 0.76.

By the alternating series test, this estimate is accurate to within an error of less than

$$\frac{1}{2880} \cong 0.000347 < 0.001.$$

Exercises

1. Find the approximate value of $x > 0$, if $e^{-2x} = 3x^2$.

2. Use power series to solve the initial-value problem $y' = -3y,\ y(0) = 5$.

3. Evaluate $\int_0^1 \sin x^2\,dx$ to within an error of 0.001.