Sequences and Series

A sequence of real numbers is a function $a : N \to R$.

The sequence is denoted by $\{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ or $\{a_n\}$

For example, the expression $\{2n\}$ denotes the sequence $\{2, 4, 6, \cdots\}$

The number a_n is called the general term of the sequence $\{a_n\}$.

Convergent sequence: We say that the sequence $\{a_n\}$ converges to L if $\lim\limits_{n\to\infty} a_n = L$, otherwise it diverges.

Example 1: Determine whether the sequence converges or diverges.

1. $\left\{\dfrac{n^2+1}{(n+1)^2}\right\}$ 2. $\left\{\dfrac{n+17}{\sqrt{2n^2+3n}}\right\}$ 3. $\left\{\dfrac{e^n}{n^3}\right\}$

1. $\lim\limits_{n\to\infty} \dfrac{n^2+1}{(n+1)^2} = \lim\limits_{n\to\infty} \dfrac{2n}{2(n+1)} = \lim\limits_{n\to\infty} \dfrac{2}{2} = 1$ converges to 1

2. $\lim\limits_{n\to\infty} \dfrac{n+17}{\sqrt{2n^2+3n}} = \lim\limits_{n\to\infty} \dfrac{1+\frac{17}{n}}{\sqrt{2+\frac{3}{n}}} = \dfrac{1}{\sqrt{2}}$ converges to $\dfrac{1}{\sqrt{2}}$

3. $\lim\limits_{n\to\infty} \dfrac{e^n}{n^3} = \lim\limits_{n\to\infty} \dfrac{e^n}{3n^2} = \lim\limits_{n\to\infty} \dfrac{e^n}{6n} = \lim\limits_{n\to\infty} \dfrac{e^n}{6} = \infty$ diverges

Limits that arise frequently

1. $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$

2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim\limits_{n\to\infty} x^{\frac{1}{n}} = 1 \;;\; (x > 0)$

4. $\lim\limits_{n\to\infty} x^n = 0 \;;\; (|x| < 1)$

5. $\lim\limits_{n\to\infty} \left(1+\dfrac{x}{n}\right)^n = e^x$

6. $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$

Example 2: Determine whether the sequence converges or diverges.

1. $\left\{\frac{1+\ln n}{n}\right\}$ 2. $\left\{\frac{1}{(0.6)^n}\right\}$ 3. $\left\{\left(\frac{x^n}{2n+1}\right)^{1/n}\right\}$

4. $\left\{\sqrt[n]{n^2+n}\right\}$ 5. $\left\{\ln\left(\frac{n-2}{n}\right)^n\right\}$ 6. $\left\{\frac{n!}{10^n}\right\}$

1. $\lim\limits_{n\to\infty}\frac{1+\ln n}{n}=\lim\limits_{n\to\infty}\frac{1}{n}+\lim\limits_{n\to\infty}\frac{\ln n}{n}=0$ converges to 0

2. $\lim\limits_{n\to\infty}\frac{1}{(0.6)^n}=\frac{1}{\lim\limits_{n\to\infty}(0.6)^n}=\infty$ diverges

3. $\lim\limits_{n\to\infty}\left(\frac{x^n}{2n+1}\right)^{1/n}=\frac{\lim\limits_{n\to\infty}x}{\lim\limits_{n\to\infty}\sqrt[n]{2n+1}}=\frac{x}{1}=x$ converges to x

4. $\lim\limits_{n\to\infty}\sqrt[n]{n^2+n}=\lim\limits_{n\to\infty}\sqrt[n]{n(n+1)}=\lim\limits_{n\to\infty}\sqrt[n]{n}\times\lim\limits_{n\to\infty}\sqrt[n]{n+1}=1$ converges to1

5. $\lim\limits_{n\to\infty}\ln\left(\frac{n-2}{n}\right)^n=\ln\lim\limits_{n\to\infty}\left(1+\frac{-2}{n}\right)^n=\ln e^{-2}=-2$ converges to -2

6. $\lim\limits_{n\to\infty}\frac{n!}{10^n}=\frac{1}{\lim\limits_{n\to\infty}\frac{10^n}{n!}}=\frac{1}{0}=\infty$ diverges

H.W

Determine whether the sequence converges or diverges.

1. $\left\{\frac{3+\ln n^n}{n^2}\right\}$ 2. $\left\{\left(\frac{3}{n}\right)^{\frac{1}{n}}\right\}$

3. $\left\{\left(\frac{n+5}{n}\right)^n\right\}$ 4. $\{\sqrt[n]{4^n n}\}$

5. $\left\{\frac{n^5+2n}{3n^4+n^2}\right\}$ 6. $\left\{\frac{8^{n+1}}{n!}\right\}$

Infinite Series

An infinite series is given by the terms of an infinite sequence, added together. For example, we could take the infinite sequence $\{2n\} = \{2, 4, 6, \cdots\}$

Then the corresponding example of an infinite series would be given by all of these terms added together $2 + 4 + 6 + \cdots$, so we have $\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + \cdots$

Geometric Series

A geometric series is any series that can be written in the form,

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots + ar^n + \cdots$$

1. If $|r| < 1$ then the geometric series converges to the sum $s_n = \dfrac{a}{1-r}$
2. If $|r| \geq 1$ then the geometric series diverges to ∞ .

For example the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$

converges because $r = \dfrac{1}{2} < 1$ and the sum is $s_n = \dfrac{1/2}{1-(1/2)} = 1.$

The series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$ diverges to ∞ because $r = \dfrac{3}{2} > 1$

The p – series

If p is a real constant, then the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$

1. converges if $p > 1$.   2. diverges if $p \leq 1$.

For example the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges because $p = 2 > 1$

The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges to ∞ because $p = \dfrac{1}{2} < 1$

Tests for converges of series

1. Integral Test

Let the function $f(x) = a_n(x)$ be continuous, positive and decreasing then the series $\sum_{n=1}^{\infty} a_n$ and the integral $\int_1^{\infty} f(x)\,dx$ both converge or both diverge.

Example 1: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ 2. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ 3. $\sum_{n=1}^{\infty} \frac{1}{n^2-n-3}$

1. $\int_1^{\infty} \frac{dx}{x^2+1} = \tan^{-1} x \Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$ converge

2. $\int_1^{\infty} \frac{xdx}{x^2+1} = \frac{1}{2}\ln|x^2+1|\Big|_1^{\infty} = \infty$ diverge

3. $\frac{1}{x^2-x-2} = \frac{A}{x-2} + \frac{B}{x+1}$: $x = 2 \Rightarrow A = \frac{1}{3}$ and $x = -1 \Rightarrow B = -\frac{1}{3}$

$$\int_1^{\infty} \frac{dx}{x^2-x-2} = \frac{1}{3}\int_1^{\infty}\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx = \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\Bigg|_1^{\infty}$$

$$= \frac{1}{3}\ln\left|\frac{1-\frac{2}{x}}{1+\frac{1}{x}}\right|\Bigg|_1^{\infty} = \frac{1}{3}\ln 2 \quad \text{converge}$$

2. Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with non – negative terms and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L$.

If 1. $L < 1$ then the series converges

2. $L > 1$ then the series diverges

3. $L = 1$ then this test is inconclusive

Example 2: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty} \frac{n^3}{3^n}$ 2. $\sum_{n=1}^{\infty} \frac{n!}{2^n}$ 3. $\sum_{n=1}^{\infty} \frac{3}{2n+5}$ 4. $\sum_{n=1}^{\infty} \frac{5^n}{n\,3^n}$

1. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3} = \frac{1}{3} \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3$

$$= \frac{1}{3} \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3} < 1 \quad \text{converges}$$

2. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{2^{n+1}} \times \frac{2^n}{n!}$

$$= \lim_{n\to\infty} \frac{(n+1) \times n!}{2^n \times 2} \times \frac{2^n}{n!} = \lim_{n\to\infty} \frac{n+1}{2} = \infty > 1 \quad \text{diverges}$$

3. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3}{2(n+1)+5} \times \frac{2n+5}{3}$

$$= \lim_{n\to\infty} \frac{2n+5}{2n+7} = 1. \quad \text{The test is inconclusive .}$$

So we try another test $\int_1^{\infty} \frac{3dx}{2x+5} = \frac{3}{2} \ln|2x+5| \Big|_1^{\infty} = \infty$ diverges

4. $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(n+1)\,3^{n+1}} \times \frac{n\,3^n}{5^n}$

$$= \lim_{n\to\infty} \frac{5n}{3(n+1)} = \frac{5}{3} > 1 \quad \text{diverges}$$

3. Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with non − negative terms and $\lim_{n\to\infty} \sqrt[n]{a_n} = L$.

If 1. $L < 1$ then the series converges

2. $L > 1$ then the series diverges

3. $L = 1$ then this test is inconclusive

Example 3: Determine whether the series converges or diverges.

1. $\sum_{n=1}^{\infty}\left(\frac{3n+2}{4n+1}\right)^n$ 2. $\sum_{n=1}^{\infty}\left(\frac{1}{n}\right)^n$ 3. $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$ 4. $\sum_{n=1}^{\infty}\left(\frac{n-1}{n}\right)^n$

1. $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{3n+2}{4n+1}=\frac{3}{4}<1$ converges

2. $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{1}{n}=0<1$ converges

3. $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{2}{\sqrt[n]{n^2}}=\frac{2}{\lim_{n\to\infty}\sqrt[n]{n}}=\frac{2}{1}=2>1$ diverges

4. $\lim_{n\to\infty}\sqrt[n]{a_n}=\lim_{n\to\infty}\frac{n-1}{n}=1.$ The test is inconclusive .

So we try another test $\lim_{n\to\infty}\left(\frac{n-1}{n}\right)^n=\lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n=e^{-1}\neq 0$ diverges

H.W

Find the sum of the following series:

1. $\sum_{n=1}^{\infty}\frac{2^n}{5^n}$ 2. $\sum_{n=1}^{\infty}\frac{5^n}{3^{2n}}$ 3. $\sum_{n=1}^{\infty}\frac{4}{3^{n+1}}$

Determine whether the series converges or diverges. Give reasons for your answers.

4. $\sum_{n=1}^{\infty}\left(\frac{n}{2n-1}\right)^n$ 5. $\sum_{n=1}^{\infty}\frac{1}{n^2\sqrt{n}}$ 6. $\sum_{n=1}^{\infty}\frac{(n-1)!}{n\,!}$

7. $\sum_{n=1}^{\infty}\frac{11^n}{3^{2n}}$ 8. $\sum_{n=1}^{\infty}ne^{-n^2}$ 9. $\sum_{n=1}^{\infty}\left(\frac{3n+4}{2n}\right)^{-n}$