Sequences

A sequence of real numbers is a function $a: N \to R$. $S: N \to R$

The sequence is denoted by $\{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ or $\{a_n\}$

For example, the expression $\{2n\}$ denotes the sequence $\{2, 4, 6, \cdots\}$

The number a_n is called the general term of the sequence $\{a_n\}$.

convergent sequence: We say that the sequence $\{a_n\}$ converges to L if $\lim_{n\to\infty} a_n = L$, otherwise it diverges.

Example 1: Determine whether the sequence converges or diverges.

1. $\left\{\dfrac{n^2+1}{(n+1)^2}\right\}$ 2. $\left\{\dfrac{n+17}{\sqrt{2n^2+3n}}\right\}$ 3. $\left\{\dfrac{e^n}{n^3}\right\}$

4. $\left\{\sqrt{n+4} - \sqrt{n}\right\}$ 5. $\left\{\sqrt{n^2+5n} - n\right\}$

1. $\lim_{n\to\infty} \dfrac{n^2+1}{(n+1)^2} = \lim_{n\to\infty} \dfrac{2n}{2(n+1)} = \lim_{n\to\infty} \dfrac{2}{2} = 1$ converges to 1

2. $\lim_{n\to\infty} \dfrac{n+17}{\sqrt{2n^2+3n}} = \lim_{n\to\infty} \dfrac{1+\frac{17}{n}}{\sqrt{2+\frac{3}{n}}} = \dfrac{1}{\sqrt{2}}$ converges to $\dfrac{1}{\sqrt{2}}$

3. $\lim_{n\to\infty} \dfrac{e^n}{n^3} = \lim_{n\to\infty} \dfrac{e^n}{3n^2} = \lim_{n\to\infty} \dfrac{e^n}{6n} = \lim_{n\to\infty} \dfrac{e^n}{6} = \infty$ diverges

4. $\lim_{n\to\infty} \left(\sqrt{n+4} - \sqrt{n}\right) = \lim_{n\to\infty} \left(\sqrt{n+4} - \sqrt{n}\right) \times \dfrac{\left(\sqrt{n+4}+\sqrt{n}\right)}{\left(\sqrt{n+4}+\sqrt{n}\right)}$

$$= \lim_{n\to\infty} \frac{\cancel{n}+4-\cancel{n}}{\left(\sqrt{n+4}+\sqrt{n}\right)} = 0 \text{ converges to } 0 \qquad \frac{4}{\infty} = 0$$

5. $\lim_{n\to\infty} \left(\sqrt{n^2+5n} - n\right) \times \dfrac{\sqrt{n^2+5n}+n}{\sqrt{n^2+5n}+n} = \lim_{n\to\infty} \dfrac{n^2+5n-n^2}{\sqrt{n^2+5n}+n}$

$$= \lim_{n\to\infty} \frac{5n}{\sqrt{n^2+5n}+n} = \lim_{n\to\infty} \frac{5}{\sqrt{1+\frac{5}{n}}+1}$$

$$= \frac{5}{2} \text{ converges to } \frac{5}{2}$$

Limits that arise frequently

1. $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = 0$ 2. $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

3. $\lim\limits_{n\to\infty} x^{1/n} = 1 \;\; ; \;\; (x > 0)$ 4. $\lim\limits_{n\to\infty} x^n = 0 \;\; ; \; (|x| < 1)$ حفظ

5. $\lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n = e^x$ 6. $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$

Calculation of the limits

1. $\lim\limits_{n\to\infty} \dfrac{\ln n}{n} = \lim\limits_{n\to\infty} \dfrac{\frac{1}{n}}{1} = 0$

2. Let $a = \lim\limits_{n\to\infty} \sqrt[n]{n}$ then $\ln a = \ln \lim\limits_{n\to\infty} \sqrt[n]{n}$

$$\ln a = \ln \lim_{n\to\infty} (n)^{1/n}$$

$$\ln a = \lim_{n\to\infty} \ln(n)^{1/n}$$

$$\ln a = \lim_{n\to\infty} \frac{\ln n}{n} = 0$$

$$a = e^0 = 1$$

$$\text{So } \lim_{n\to\infty} \sqrt[n]{n} = 1$$

5. Let $a = \lim\limits_{n\to\infty} \left(1 + \dfrac{x}{n}\right)^n$ then $\ln a = \lim\limits_{n\to\infty} \ln\left(1 + \dfrac{x}{n}\right)^n$

$$\ln a = \lim_{n\to\infty} n \ln\left(1 + \frac{x}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1 + \frac{x}{n}\right)}{\frac{1}{n}}$$

$$\ln a = \lim_{n\to\infty} \frac{\dfrac{1}{\left(1 + \frac{x}{n}\right)} \cdot \dfrac{-x}{n^2}}{\dfrac{-1}{n^2}} = x$$

$$a = e^x$$

$$\text{So } \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

Example 2: Determine whether the sequence converges or diverges.

1. $\left\{\dfrac{1+\ln n}{n}\right\}$ 2. $\left\{\dfrac{1}{(0.6)^n}\right\}$ 3. $\left\{\left(\dfrac{x^n}{2n+1}\right)^{1/n}\right\}$

4. $\left\{\sqrt[n]{n^2+n}\right\}$ 5. $\left\{\ln\left(\dfrac{n-2}{n}\right)^n\right\}$ 6. $\left\{\dfrac{n!}{10^n}\right\}$

1. $\displaystyle \lim_{n\to\infty}\frac{1+\ln n}{n} = \lim_{n\to\infty}\frac{1}{n} + \lim_{n\to\infty}\frac{\ln n}{n} = 0$, converges to 0

2. $\displaystyle \lim_{n\to\infty}\frac{1}{(0.6)^n} = \frac{1}{\lim\limits_{n\to\infty}(0.6)^n} = \infty$, diverges $\frac{1}{0} = \infty$

3. $\displaystyle \lim_{n\to\infty}\left(\frac{x^n}{2n+1}\right)^{1/n} = \frac{\lim\limits_{n\to\infty} x}{\lim\limits_{n\to\infty}\sqrt[n]{2n+1}} = \frac{x}{1} = x$ converges to x

4. $\displaystyle \lim_{n\to\infty}\sqrt[n]{n^2+n} = \lim_{n\to\infty}\sqrt[n]{n(n+1)} = \lim_{n\to\infty}\sqrt[n]{n} \times \lim_{n\to\infty}\sqrt[n]{n+1} = \underline{1}$ converges to1

5. $\displaystyle \lim_{n\to\infty}\ln\left(\frac{n-2}{n}\right)^n = \ln\lim_{n\to\infty}\left(1+\frac{-2}{n}\right)^n = \ln e^{-2} = -2$ converges to -2

6. $\displaystyle \lim_{n\to\infty}\frac{n!}{10^n} = \frac{1}{\lim\limits_{n\to\infty}\frac{10^n}{n!}} = \frac{1}{0} = \infty$ diverges $4! = 4\cdot 3\cdot 2\cdot 1$

Exercises

Determine whether the sequence converges or diverges.

1. $\left\{\dfrac{3+\ln n^n}{n^2}\right\}$ ✓ 2. $\left\{\left(\dfrac{3}{n}\right)^{1/n}\right\}$ ✓3. $\left\{\left(\dfrac{n+5}{n}\right)^n\right\}$

4. $\left\{\sqrt[n]{4^n n}\right\}$ ✓ 5. $\left\{\dfrac{n^5+2n}{3n^4+n^2}\right\}$ ✓ 6. $\left\{\dfrac{8^{n+1}}{n!}\right\}$

7. $\left\{\dfrac{e^n+n^2}{e^n-2n^2}\right\}$ 8. $\left\{\sqrt{n(n+2)}-n\right\}$ 9. $\left\{\left(\dfrac{x^n}{2n+1}\right)^{1/n}\right\}$

10. $\left\{\dfrac{3n^2-\ln n}{n^2+3n^{3/2}}\right\}$ 11. $\left\{\left(\dfrac{n-3}{n}\right)^n\right\}$

H.W.