Vector Space

Definition 4.2.1 Let V be a set on which two operations (**vector addition** and **scalar multiplication**) are defined. If the listed axioms are satisfied for every $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in V and scalars c and d, then V is called a **vector space** (over the reals $\mathbb{R}$).

1. Addition:

   (a) $\mathbf{u} + \mathbf{v}$ is a vector in V (*closure under addition*).

   (b) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (*Commutative property of addition*).

   (c) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ (*Associative property of addition*).

   (d) There is a **zero vector** $\mathbf{0}$ in V such that for every $\mathbf{u}$ in V we have $(\mathbf{u} + \mathbf{0}) = \mathbf{u}$ (*Additive identity*).

   (e) For every $\mathbf{u}$ in V, there is a vector in V denoted by $-\mathbf{u}$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (*Additive inverse*).

2. Scalar multiplication:

   (a) $c\mathbf{u}$ is in V (*closure under scalar multiplication*0.

   (b) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ (*Distributive propertyof scalar mult.*).

   (c) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ (*Distributive property of scalar mult.*).

   (d) $c(d\mathbf{u}) = (cd)\mathbf{u}$ (*Associate property of scalar mult.*).

   (e) $1(\mathbf{u}) = \mathbf{u}$ (*Scalar identity property*).

Example

Let $V = \{(x, \frac{1}{2}x) : x\ real\ number\}$ with standard operations. Is it a vector space. Justify your answer.

Solution: Yes, V is a vector space. We check all the properties in , one by one:

1. Addition:
   (a) For real numbers x, y, We have
   $$\left(x, \frac{1}{2}x\right) + \left(y, \frac{1}{2}y\right) = \left(x + y, \frac{1}{2}(x + y)\right).$$
   So, V is closed under addition.
   (b) Clearly, addition is closed under addition.
   (c) Clearly, addition is associative.
   (d) The element $\mathbf{0} = (0, 0)$ satisfies the property of the zero element.
   (e) We have $-\left(x, \frac{1}{2}x\right) = \left(-x, \frac{1}{2}(-x)\right)$ .So, every element in V has an additive inverse.
2. Scalar multiplication:
   (a) For a scalar c, we have
   $$c\left(x, \frac{1}{2}x\right) = \left(cx, \frac{1}{2}cx\right).$$
   So, V is closed under scalar multiplication.
   (b) The distributivity $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ works for $\mathbf{u}, \mathbf{v}$ in V.
   (c) The distributivity $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ works, for $\mathbf{u}$ in V and scalars c, d.
   (d) The associativity $c(d\mathbf{u}) = (cd)\mathbf{u}$ works.
   (e) Also $1\mathbf{u} = \mathbf{u}$.

Theorem

Let V be vector space over the reals $\mathbb{R}$ and $\mathbf{v}$ be an element in V. Also let c be a scalar. Then,

1. $0\mathbf{v} = \mathbf{0}$.
2. $c\mathbf{0} = \mathbf{0}$.
3. If $c\mathbf{v} = \mathbf{0}$, then either $c = 0$ or $\mathbf{v} = \mathbf{0}$.
4. $(-1)\mathbf{v} = -\mathbf{v}$.

Example

- Let V be the set of all fifth-degree polynomials with standard operations. Is it a vector space. Justify your answer.

  Solution: In fact, V is not a vector space. Because V is not closed under addition(axiom (1a) of definition fails): $f = x^5 + x - 1$ and $g = -x^5$ are in V but $f + g = (x^5 + x - 1) - x^5 = x - 1$ is not in V.

- Let $V = \{(x, y) : x \geq 0, y \geq 0\}$ with standard operations. Is it a vector space. Justify your answer.

  Solution: In fact, V is not a vector space. Not every element in V has an addditive inverse (axiom (1e) of fails): $-(1, 1) = (-1, -1)$ is not in V.

Subspaces of Vector Spaces

Definition

A nonempty subset W of a vector space V is called a subspace of V if W is a vector space under the operations addition and scalar multiplication defined in V.

Example

Let $W = \{(x, 0) : x \text{ is real number}\}$. Then $W \subseteq \mathbb{R}^2$. (*The notation* $\subseteq$ *reads as 'subset of'.*) It is easy to check that W is a subspace of $\mathbb{R}^2$.

Theorem

Suppose V is a vector space over $\mathbb{R}$ and $W \subseteq V$ is a **nonempty** subset of V. Then W is a subspace of V if and only if the following two closure conditions hold:

1. If $\mathbf{u}, \mathbf{v}$ are in W, then $\mathbf{u} + \mathbf{v}$ is in W.
2. If $\mathbf{u}$ is in W and c is a scalar, then $c\mathbf{u}$ is in W.

Example

Let H = $\left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} a \text{ and b are real} \right\}$. Show that H is a subspace of R^3.

Solution:

Verify properties a, b and c of the definition of a subspace.

a. The zero vector of R^3 is in H (let a =0 and b =0).
b. Adding two vectors in H always produces another vector whose second entry is and therefore the sum of two vectors in H is also in H. (H is closed under addition)
c. Multiplying a vector in *H* by a scalar produces another vector in *H* (*H* is closed under scalar multiplication).

Since properties *a*, *b*, and c hold, *V* is a subspace of R^3 .

Example

Is $H = \left\{ \begin{matrix} x \\ x+1 \end{matrix} : x \text{ is real} \right\}$ a subspace of R^2 ?

Solution:

For H to be a subspace of R^2 , all three properties must hold Property (a) fails.

Therefore H is not a subspace of R^2 .

Another way to show that H is not a subspace of R^2 :

Let $u = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, then $u + v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ .So property (b) fails and so H is not a subspace of R^2.

Definition

. A vector $\mathbf{v}$ in a vector space V is called a **linear combination** of vectors $\mathbf{u_1}, \mathbf{u_2}, \ldots, \mathbf{u_k}$ in V if $\mathbf{v}$ can be written in the form

$$\mathbf{v} = c_1\mathbf{u_1} + c_2\mathbf{u_2} + \cdots + c_k\mathbf{u_k},$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Definition

. Let V be a vector space over $\mathbb{R}$ and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. We say that S is a **spanning set** of V if every vector $\mathbf{v}$ of V can be written as a liner combination of vectors in S. In such cases, we say that S **spans** V.

Definition

. Let V be a vector space over $\mathbb{R}$ and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$ be a subset of V. Then the **span of** S is the set of all linear combinations of vectors in S,

$$span(S) = \{c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} : c_1, c_2, \ldots, c_k \text{ are scalars}\}.$$

1. The span of S is denoted by $span(S)$ as above or $span\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k}\}$.
2. If $V = span(S)$, then say V is spanned by S or S spans V.

Theorem:

If $v_1, \ldots, v_n$ are in a vector space V, then Span$\{v_1, \ldots, v_n\}$ is a subspace of V.

Example

Is $V = \{(a + 2b, 2a - 3b) : a \text{ and } b \text{ are real}\}$ a subspace of R^2 ?

Solution: Write vectors in V in column form:

$$\begin{bmatrix} a + 2b \\ 2a - 3b \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

So V =Span$\{v_1, v_2\}$ and therefore V is a subspace by Theorem 1.

Example

Is $H = \left\{ \begin{bmatrix} a + 2b \\ a + 1 \\ a \end{bmatrix} a \text{ and } b \text{ are real} \right\}$ a subspace of R^3 ?

Solution:

0 is not in H since $a = b = 0$ or any other combination of values for a and b does not produce the zero vector.

So property fails to hold and therefore H is not a subspace of R^3.

Example

Is the set H of all matrices of the form $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix}$ a subspace of $M_{2\times2}$?

Solution:

Since $\begin{bmatrix} 2a & b \\ 3a+b & 3b \end{bmatrix} = \begin{bmatrix} 2a & 0 \\ 3a & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ b & 3b \end{bmatrix}$

Therefore H =Span $\left\{\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 3 \end{bmatrix}\right\}$ and so H is a subspace of $M_{2\times2}$.

Definition

Let V be a vector space. A set of elements (vectors) $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ is said to be **linearly independent** if the equation

$$c_1\mathbf{v_1} + c_2\mathbf{v_2} + \cdots + c_k\mathbf{v_k} = \mathbf{0}$$

has only trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

We say S is **linearly dependent**, if S in not linearly independent. (*This means, that S is said to be linearly dependent, if there is at least one nontrivial (i.e. nonzero) solutions to the above equation.*)

Example

Let $S = \{(6,2,1), (-1,3,2)\}$. Determine, if S is linearly independent or dependent?

Solution: Let

$$c(6,2,1) + d(-1,3,2) = (0,0,0).$$

If this equation has only trivial solutions, then it is linealry independent. This equaton gives the following system of linear equations:

$$\begin{aligned} 6c \quad -d \quad &= 0 \\ 2c \quad +3d \quad &= 0 \\ c \quad +2d \quad &= 0 \end{aligned}$$

The augmented matrix for this system is

$$\begin{bmatrix} 6 & -1 & 0 \\ 2 & 3 & 0 \\ 1 & 2 & 0 \end{bmatrix}. \quad its\ gauss - Jordan\ form: \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So, $c = 0, d = 0$. The system has only trivial (i.e. zero) solution. We conclude that S is linearly independent.

Exercise

Let

$$S = \{(1,0,0), (0,4,0), (0,0,-6), (1,5,-3)\}.$$

Determine, if S is linearly independent or dependent?

Basis

Definition

Let V be a vector space and $S = \{\mathbf{v_1}, \mathbf{v_2}, \ldots \mathbf{v_k}\}$ be a set of elements (vectors)in V. We say that S is a **basis** of V if

1. S spans V and
2. S is linearly independent.

Example

standard basis of $\mathbb{R}^n$.

1. Consider the vector space $\mathbb{R}^2$. Write

$$\mathbf{e_1} = (1, 0), \mathbf{e_2} = (0, 1).$$

Then, $\mathbf{e_1}, \mathbf{e_2}$ form a basis of $\mathbb{R}^2$.

2. Consider the vector space $\mathbb{R}^3$. Write

$$\mathbf{e_1} = (1, 0, 0), \mathbf{e_2} = (0, 1, 0), \mathbf{e_2} = (0, 0, 1).$$

Then, $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ form a basis of $\mathbb{R}^3$.

3. More generally, consider vector space $\mathbb{R}^n$. Write

$$\mathbf{e_1} = (1, 0, \ldots, 0), \mathbf{e_2} = (0, 1, \ldots, 0), \ldots, \mathbf{e_n} = (0, 0, \ldots, 1).$$

Then, $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \ldots, \mathbf{e_n}$ form a basis of $\mathbb{R}^n$. The proof will be similar to the above proof. This basis is called the **standard basis** of $\mathbb{R}^n$.

4.

Explain, why the set

$$S = \{(2,1,-2), (-2,-1,2), (4,2,-4)\}$$

is not a basis of $\mathbb{R}^3$?

Solution: Note

$$(4,2,-4) = (2,1,-2) - (-2,-1,2)$$

OR

$$(2,1,-2) - (-2,-1,2) - (4,2,-4) = (0,0,0).$$

So, these three vectors are linearly dependent. So, S is not a basis of $\mathbb{R}^3$.

5.

Explain, why the set

$$S = \{6x - 3, 3x^2, 1 - 2x - x^2\}$$

is not a basis of $\mathbb{P}_2$?

Solution: Note

$$1 - 2x - x^2 = -\frac{1}{3}(6x - 3) - \frac{1}{3}(3x^2)$$

OR

$$(1 - 2x - x^2) + \frac{1}{3}(6x - 3) + \frac{1}{3}(3x^2) = \mathbf{0}.$$

So, these three vectors are linearly dependent. So, S is not a basis of $\mathbb{P}_2$.