# 3D Stress Tensors

## 3D Stress Tensors, Eigenvalues and Rotations

Recall that we can think of an n x n matrix $M_{ij}$ as a transformation matrix that transforms a vector $x_i$ to give a new vector $y_j$ (first index = row, second index = column), e.g. the equation $Mx = y$. We define x to be an eigenvector of M if there exists a scalar λ such that

$$Mx = \lambda x \qquad (34)$$

The value(s) λ are called the eigenvalues of M. We can λnd the eigenvalues simply, as follows. First we can infer that

$$Mx - \lambda x = 0 \qquad (35)$$

where 0 is a zero (null) vector. Then we can simplify this by introducing the Identity matrix I of the same size as M

$$(M - \lambda I)x = 0 \qquad (36)$$

This implies that $\det(M - \lambda I) = 0$, since the only solution is found where the size of

$M - \lambda I$

is zero, where det signifies the determinant. We are interested in solving this problem for 3x3 matrices, that is for 3D states of stress (or, as we shall see later, strain). You will recall the checkerboard pattern for finding a determinant of a matrix A;

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix} \qquad (37)$$

We find the determinant of A by multiplying each of the terms in any row or column by (i) the corresponding sign from the checkerboard pattern and (ii) the determinant of the 2x2 matrix left out once that term's row and column have been removed. e.g.

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \qquad (38)$$

So if we have a square symmetric stress matrix $\sigma_{ij}$ then its eigenvalues will be given by

$$|\sigma - \lambda I| = +(\sigma_{11} - \lambda)\left\{(\sigma_{22} - \lambda)(\sigma_{33} - \lambda) - \sigma_{23}^2\right\} - \sigma_{12}\left\{\sigma_{12}(\sigma_{33} - \lambda) - \sigma_{23}\sigma_{13}\right\} + \sigma_{13}\left\{\sigma_{12}\sigma_{23} - (\sigma_{22} - \lambda)\sigma_{13}\right\} \quad (39)$$

This is a cubic in. λ For many cases, many of the indices will be zero and the cubic will be easy to solve. Typically, the first root is found by inspection (e.g. $\lambda = 2$ is a root), at which point the problem can be reduced to a quadratic by substitution and the remaining roots found trivially.

To find the eigenvectors we then solve the equation $(\sigma\text{-}\lambda I)\ x = 0$ for each of the n eigenvalues in turn. For a 3x3 (square) symmetric (stress) matrix, this will produce three linearly independent eigenvectors. Each eigenvector will be scale-independent, since if x is an eigenvector, it is trivial to show that $\alpha x$ is also an eigenvector.

It turns out to be possible to show that in this case the eigenvalues are the principal stresses, and the eigenvectors are the equations of the axes along which the principal stresses act.

## Aside: Calculating Shear Stresses in Sections

Question. A bar of radius 50mm transmits 500kW and 6000 rpm. What is the shear stress is the bar?

Solution. We remember that Power = Torque x Angular Velocity,

$$P=T\omega \qquad (40)$$

and that the shear stress $\tau$ is related to the torque through the polar moment of inertia J and the outer radius R by

$$T/J= \tau /R \qquad (41)$$

J is given, for any section by $J = \int r^2\, dA$ so, a shaft of inner radius r and outer radius R,

$$J = \frac{1}{2}\pi(R^2 - r^2) \qquad (42)$$

Therefore, in this case,

$$J = \frac{1}{2}\pi(0.05^4) = 9.187 \times 10^{-6} m^4 \qquad (44)$$

$$T = P/\omega \qquad 5{,}000/(6000\times\left(\frac{2\pi}{60}\right) = 795.78\, Nm \qquad (45)$$

So the result is given by

$$\tau = \frac{TR}{J} = 795.78 \times \frac{0.05}{9.817} \times 10^{-6} = 4.05 MPa \qquad (46)$$

# Diagonalising matrices

In the previous section, we found the eigenvectors and eigenvalues of a matrix M. Consider the matrix of the eigenvectors X composed of each of the (column) eigenvectors x in turn, e.g. $X_{ij} = x_i$; j, and the matrix D with the corresponding eigenvalues on the leading diagonal and zeroes as the off-axis terms, e.g. $D_{ii} = \lambda_i$ and $D_{ij} = 0$ $i \neq j$. So for a 3x3 matrix M,

D=$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$. By inspection, we can see that

$$MX = XD \qquad (46)$$

because for each column, $Mx = \lambda x$.

Hence, D = $X^{-1} - MX$ (47)

A neat example of this is finding large powers of a matrix. For example,

$$M^2 = (XDx^{-1})(XDx^{-1}) = (XD^2x^{-1}) \qquad (48)$$

and so on for higher powers.

It turns out that the matrix of eigenvectors X is highly significant. Later, we will look at how to rotate a stress matrix in the general case. However, you will already be able to see that it is always possible to rotate the stress matrix using X, the rotation matrix composed of the unit eigenvectors, to produce a matrix of eigenvalues D. It turns out that this matrix is the matrix of principal stresses, i.e. that the eigenvalues of the stress matrix are the principal stresses.

# Principal Stresses in 3 Dimensions

Generalising the 2D treatment of the inclined plane to 3D, we consider an inclined plane. We take a cube with a stress state referred to the 1; 2; 3 axes, and then cut it with an inclined plane with unit normal x = $(l,\ m,\ n)$ and area A.

The components of x along the 1; 2; 3 axes are its direction cosines, that is, the cosines of the angles between x and the axes. We require that the stress _ normal to the inclined plane is a principal stress, that is that there are no shears on the inclined plane, Figure 15 First we notice that the components of σ in the 1,2 and 3 directions are σ $l$, σ $m$ and σ $n$, respectively. The areas of the triangles forming the walls of the original cube are also

$$KOL = Al \qquad JOK = Am \qquad JOL = An \qquad (49)$$

If we resolve all the forces in the 1 direction, we find that

$$\sigma Al - \sigma_{11}Al - \sigma_{21}Am - \sigma_{31}An = 0 \qquad (50)$$

$$(\sigma - \sigma_{11})l - \sigma_{21}m - \sigma_{31}n = 0 \qquad (51)$$

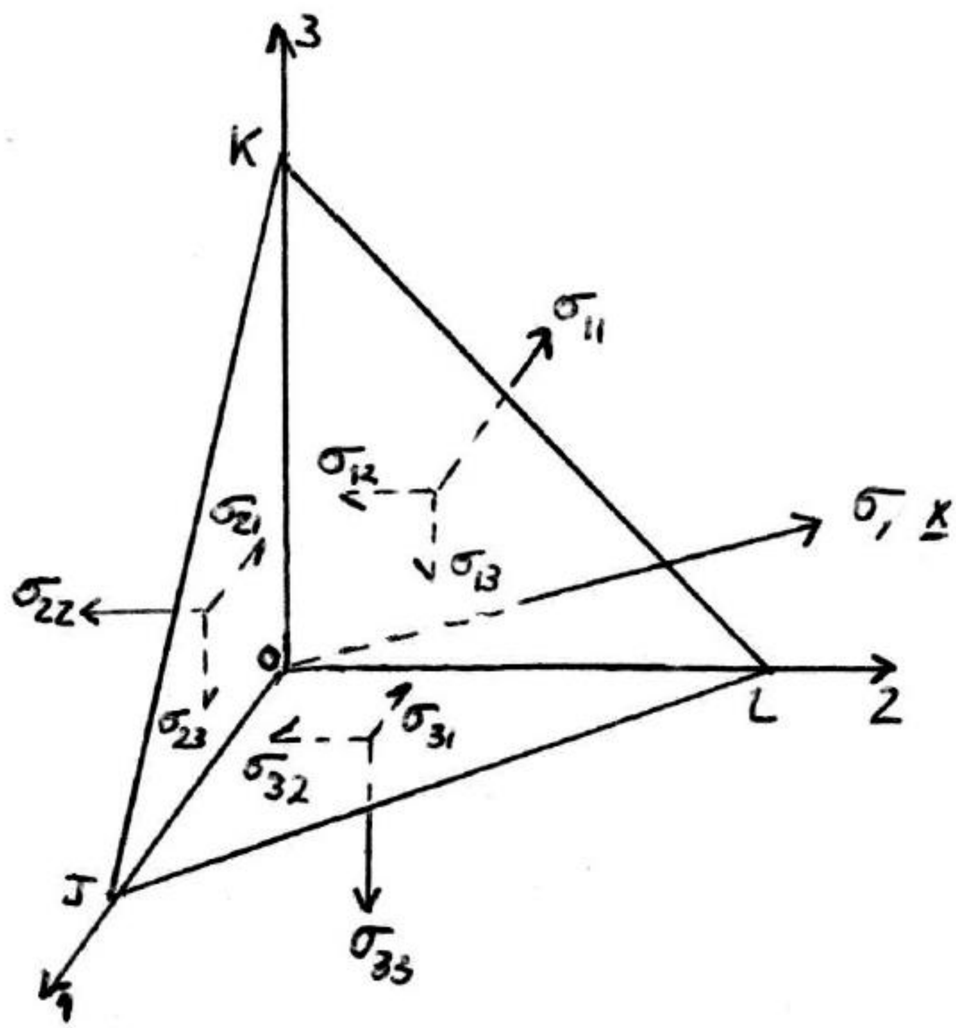


**Figure 15**: Inclined plane cut through the unit stress cube to give a principal stress σ along the x vector.

Similarly for the other two axes,

$$-\sigma_{12}l + (\sigma - \sigma_{22})m - \sigma_{32}n = 0 \qquad (52)$$
$$-\sigma_{31}l - \sigma_{32}m + (\sigma - \sigma_{33})n = 0 \qquad (53)$$

So we can write this set of simultaneous equations as (multiplying through by -1 for convenience and requiring that $\sigma_{ij} = \sigma_{ji}$)

$$\begin{pmatrix} \sigma_{11} - \sigma & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} - \sigma \sigma_{23} & \\ \sigma_{13} & \sigma_{23} & \sigma_{33} - \sigma \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = 0 \tag{54}$$

The only nontrivial solution is where the determinant of the left-hand matrix is zero, so by comparison with Equation 36 we find that the solutions for σ are the eigenvalues of the stress matrix. This is given by

$$\begin{aligned} \sigma^3 - \sigma^2(\sigma_{11} + \sigma_{22} + \sigma_{33}) + \sigma(\sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2) \\ -(\sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2) = 0 \end{aligned} \tag{55}$$

The three roots of this equation are the principal stresses. Note that the three coefficients of this equation determine the principal stresses. Therefore these coefficients cannot change under a rotation of the coordinate axes and are invariant. These invariants are

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} \tag{56}$$

$$I_2 = \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{33}\sigma_{11} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{13}^2 \tag{57}$$

$$I_3 = \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{13} - \sigma_{11}\sigma_{23}^2 - \sigma_{22}\sigma_{13}^2 - \sigma_{33}\sigma_{12}^2 \tag{58}$$

The first invariant we identify as 3 times the hydrostatic stress $\_{hyd}$, which is the average of the $\sigma_{ii}$. When we come to consider yielding, the hydrostatic stress will assume a new significance. This also implies that this hydrostatic stress is the same, in any coordinate system.

## Example: Finding principal stresses

Question. A material is subject to the following stress state. What are the principal stresses in the material?

$$\sigma = \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \text{MPa} \tag{62}$$

Solution. We need to find the eigenvalues, so;

$$\begin{vmatrix} 100-\lambda & 20 & 0 \\ 20 & 0-\lambda & 20 \\ 0 & 20 & 100-\lambda \end{vmatrix} = 0 \tag{63}$$

Which gives the cubic equation

$$(100-\lambda)\left\{-\lambda(100-\lambda)-20^2\right\} - 20\left\{20(100-\lambda)-0(20)\right\} + 0\left\{20^2-0(-\lambda)\right\} = 0 \tag{64}$$

so

$$(100-\lambda)(\lambda^2 - 100\lambda - 2 * 20^2) = 0 \tag{65}$$

which has solutions $\lambda = 100, 50 \pm 10\sqrt{17}$ MPa.

# Example: Stresses on crystal axes

Question. A single crystal is loaded as follows on its [100], [010] and [001] axes. What are the normal stresses on the (orthornormal) $\frac{1}{\sqrt{3}}[111]$, $\frac{1}{\sqrt{2}}[1\bar{1}0]$ and $\frac{1}{\sqrt{6}}[11\bar{2}]$ axes?

$$\sigma = \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \text{MPa} \tag{66}$$

Solution. First we write down the rotation matrix $a$, which is the matrix with the vectors entered in the rows;

$$a = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \tag{67}$$

Then using Equation 59, we find that

$$\sigma' = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} 100 & 20 & 0 \\ 20 & 0 & 20 \\ 0 & 20 & 100 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} \end{pmatrix}^T \tag{68}$$

So, performing the first matrix multiplication,

$$\sigma' = 20 \begin{pmatrix} \frac{6}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{6}{\sqrt{3}} \\ \frac{4}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{6}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-9}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{pmatrix} \tag{69}$$

Now we perform the second matrix multiplication to obtain;

$$\sigma' = 20 \begin{pmatrix} \frac{14}{\sqrt{9}} & \frac{4}{\sqrt{6}} & \frac{-4}{\sqrt{18}} \\ \frac{4}{\sqrt{6}} & \frac{3}{\sqrt{4}} & \frac{7}{\sqrt{12}} \\ \frac{-4}{\sqrt{18}} & \frac{7}{\sqrt{12}} & \frac{23}{\sqrt{36}} \end{pmatrix} = \begin{pmatrix} 93.3 & 32.7 & -18.9 \\ 32.7 & 30 & 40.4 \\ -18.9 & 40.4 & 76.7 \end{pmatrix} \text{MPa} \tag{70}$$

The rotated matrix is still symmetric (this must be true, by theorem). When choosing the set of new basis vectors to use care must be taken to ensure that they are orthogonal to each other and of unit length (orthonormal).

Notice how it is possible, if calculationally intensive, to find the resolved shear stress on any slip system, e.g. on the (111)[110] slip system (32.7 MPa). For plasticity, however, our consideration of yielding later in the course will force us to use, rather than the total stress tensor we have so far considered, the deviatoric stress tensor which has the hydrostatic stress subtracted. This is because the hydrostratic stress cannot cause yielding.

# Strain and Elasticity

Strain is defined as the distortion of the elemental cube, as opposed to a rigid body translation or rotation of the cube. Defining the displacement field $u = (u_1, u_2, u_3)$, we first assume that $u$ is a linear function of position $x$, and so define the deformation tensor $e$ as

$$e_{ij} = \frac{\partial u_i}{\partial x_j} \tag{74}$$

e.g. $e_{11} = \frac{\partial u_1}{\partial x_1}$, $e_{21} = \frac{\partial u_2}{\partial x_1}$, $e_{12} = \frac{\partial u_1}{\partial x_2}$, etc. However, consider the following three cases.

The first case in Figure 18, (a), $e_{12} = e_{21}$, shows a situation called *pure shear*. In (b), $e_{12} = -e_{21}$, the block is simply rotating and doesn't change shape at all. In (c) $e_{21} = 0$ and $e_{12} \neq 0$, which is called *simple shear*.

Therefore we define the strain tensor $\varepsilon$ as the symmetric part of the deformation tensor $e$, and a separate rotation tensor $\omega$ which is the antisymmetric part of $e$. Thus

$$\varepsilon_{ij} = \frac{1}{2}(e_{ij} + e_{ji}), \qquad \omega_{ij} = \frac{1}{2}(e_{ij} - e_{ji}) \tag{75}$$

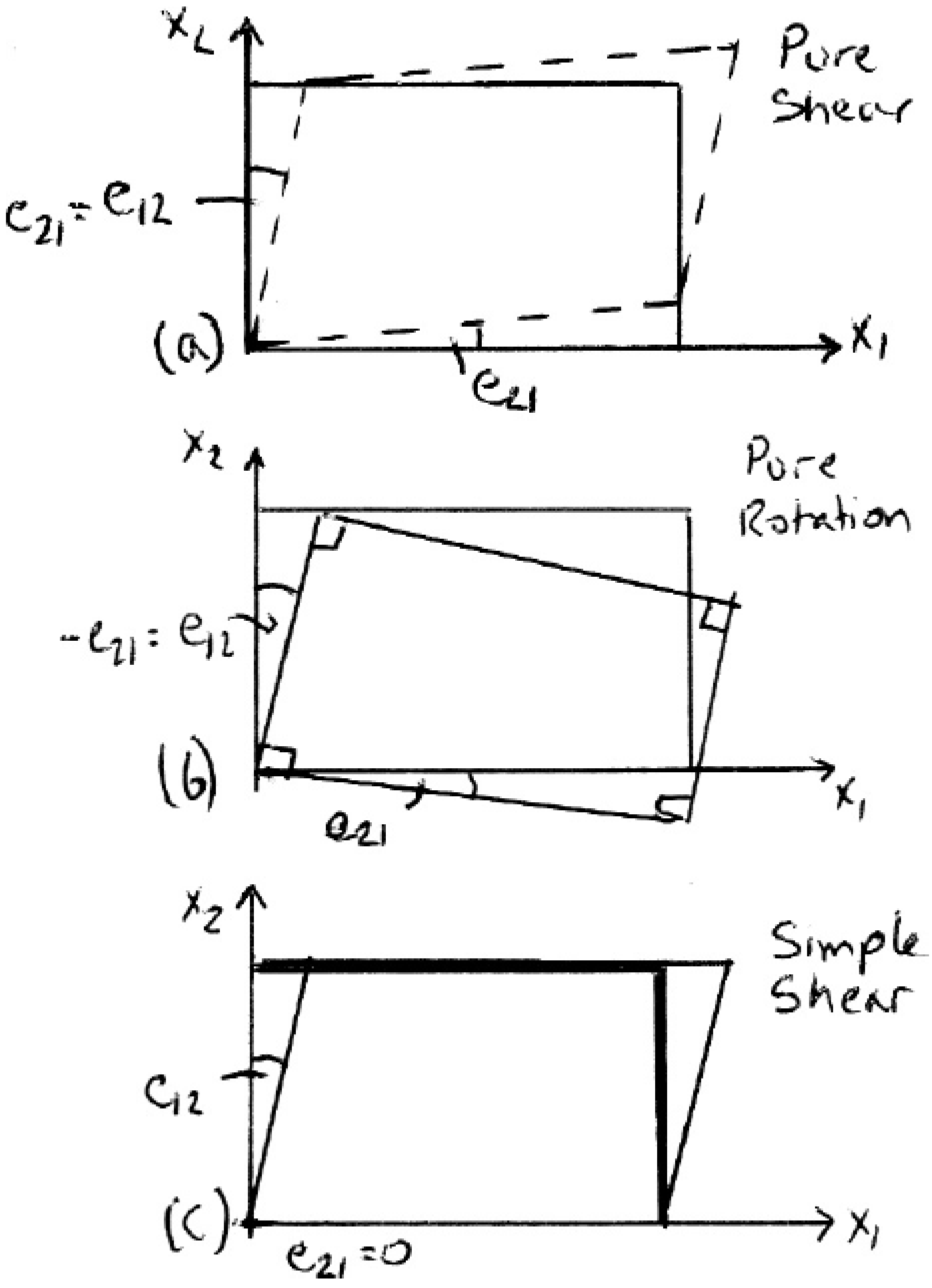


**Figure 18:** Different deformation states of an elemental cube.

Notice that e = ε + !. Thus, like stress, strain is by definition a symmetric tensor and has only 6 independent components.

There is a problem however! Conventionally, a shear strain is defined by the shear angle  produced in simple shear, below.

So in this case the tensor shear strain $\varepsilon_{12} = 1/2\,(e_{12} + e_{21}) = 1/2\,(\gamma + 0) = \gamma/2$. This problem is simply one of definition. The notation used in each case is quite standard so this is easily overcome.

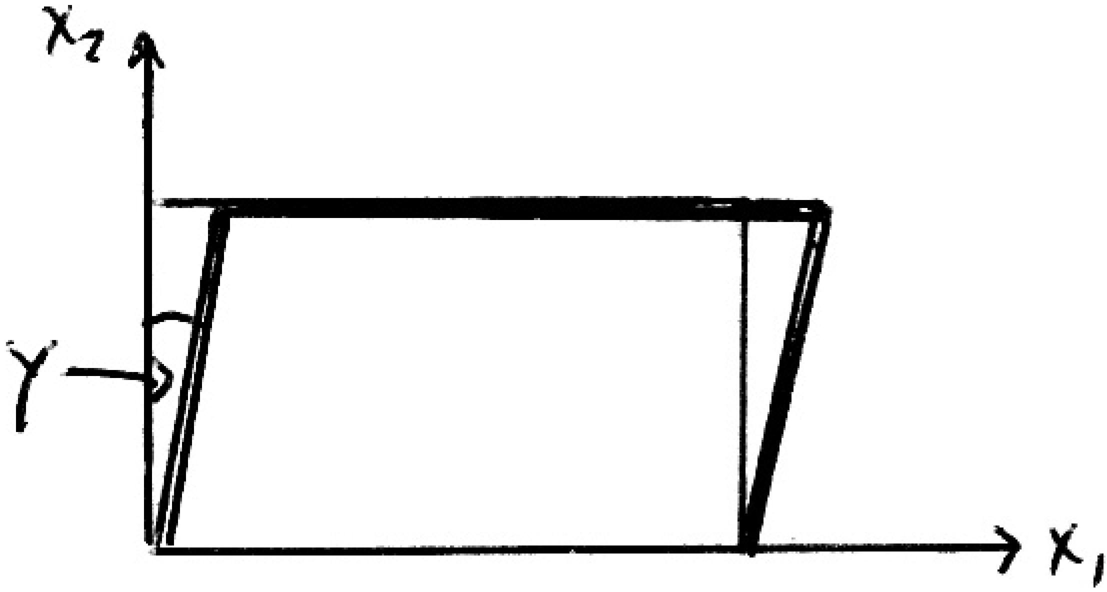


Figure 19: Definition of the simple shear strain $\gamma$ .

# Isotropic Elasticity

In general, very few materials are actually isotropic, even elastically. This is because single crystals are usually elastically and plastically anisotropic, and because most manufacturing processes produce some `uneven-ness' in the orientation distribution, or texture. Therefore we need to consider, in the general case, how to convert from stress to strain and vice-versa.

Starting with the case of simple isotropic elasticity, the basic equations are based on Hooke's Law, which in the general case gives

$$E\varepsilon_1 = \sigma_1 - \nu(\sigma_2 + \sigma_3)$$

where E is Young's Modulus and $\nu$ is Poisson's ratio. Stresses and strains are super posable, so we can combine stresses along di_erent axes. In shear, a similar equation can be written, $\tau = G\gamma$, where G is the Shear Modulus. In tensor notation, because $\varepsilon_{12} = \gamma/2$, this gives $\sigma_{ij} = 2G\varepsilon_{ij}$

However, it is easy to see that E, $v$ and G must be inter-related for an isotropic material, as follows consider a system in a state of simple shear stress,

$$\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Using Mohr's Circle, we can rotate this to find the Principal Stresses, which are that $\sigma_1 = \sigma$ and $-\sigma_2 = \sigma$

So we can find the strains along the principal axes by using Hookes Law, so $E\varepsilon_1 = \sigma - v(-\sigma + 0) = \sigma(1 + v)$, *and* $E\varepsilon_2 = -\sigma - v(\sigma + 0) = -\sigma(1 + v)$ .

We can then use Mohr's Circle for strain, in exactly the same way as for stress, Figure below Since the rotation in Mohr's circle is the same in the two cases, the strains must be equivalent and so $\varepsilon_{12}$ in the original axes is given by

$$\varepsilon_{12} = \frac{\sigma}{E}(1 + v)$$

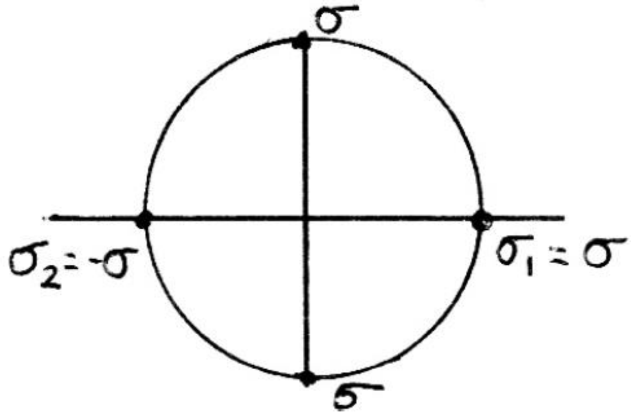


**Figure 20:** Principal stresses for applied shear $\sigma$.

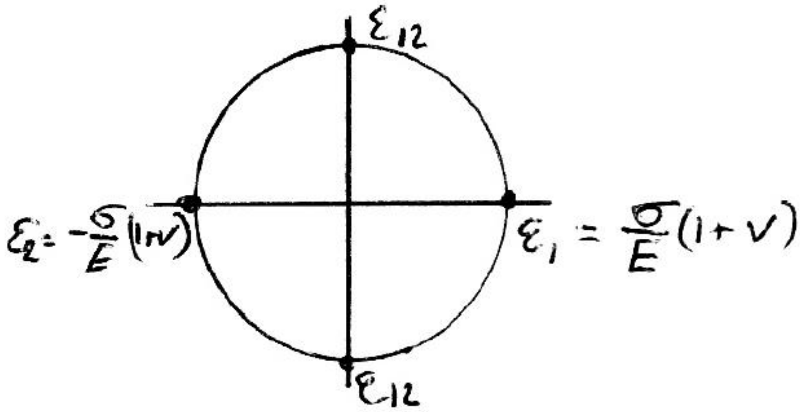


**Figure 21:** Determination of applied shear strains from the principal strains $\pm$

By comparison with Equation above we can therefore say that

$$G = \frac{E}{2(1+v)}$$

Hence for an isotropic material there are only two independent elastic constants. We can also define other related Elastic constants that are useful in di_erent stress states. The Bulk Modulus or dilatational modulus K is useful for hydrostatic stress problems, and is defined by