الدوال العقدية

المحاضرة الاولى

الدوال العقدية او الدوال المركبة (مراجعة)

(Complex functions)

1. We define complex numbers as order pairs $z = (x, y)$ of real numbers. $x\ and\ y$ Such that

x is called the real part [$Rez = x$]

y is called the imaginary part [$Imz = y$]

Ex. Let $Z = (1,3)$ then

$Rez = 1$, $Imz = 3$

الاعداد المعقدة: تعرف الاعداد المعقدة بانها ازواج مرتبة $Z = (x, y)$ حيث ان x, y عددان حقيقيان والخاضعان لعمليتي الجمع والضرب وكما يلي:

1. $z_1 + z_2 = (x_1, y_1) + (x_2, y_2)$=$(x_1 + x_2) + (\ y_1 + y_2)$.
2. $z_1.z_2 = (x_1, y_1).(x_2, y_2) = (x_1x_2 - y_1y_2), (x_1y_2 + y_1x_2)i$) .

Ex. $z_1 = (2 + 3i)$ $z_2 = (4 + 5i)$ fined

1. $z_1 + z_2$ 2. $z_1.z_2$

Sol.

1. $z_1 + z_2 = (2 + 4,3i + 5i)$= $(6 + 8i)$.or (6, 8)
2. $z_1.z_2 = (8 - 15,10i + 12i) = (-7 + 22i)$. Or (-7, 22)

Proposition: let $i = (0,1)$ then $i^2 = -1$ then $i = \sqrt{-1}$

$$i^2 = i.i = (0,1)(0,1)$$

$$= (0 - 1,0 + 0)$$

$$i^2 = (-1,0) = -1$$

Note.

$$i^3 = i^2.i = (-1).i = -i$$

$$i^4 = (-1)(-1) = 1$$

$$i^{200} = 1 \quad i^{1602} = -1 \quad i^{2003} = -i \quad i^5 = i$$

Properties. $z = x + iy = 0$ If and only if $x = 0$ & $y = 0$

Properties. $z_1 = z_2$ If and only if $(x_1 = x_2)$ & $(y_1 = y_2)$

Properties: let z_1, z_2 and z_3 be complex numbers, then

1. $z_1 + z_2 = z_2 + z_1$ Commutative law for addition

2. $z_1.z_2 = z_2.z_1$ Commutative law for multiplicative

3. $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$ Associative law

4. $z_1(z_2 + z_3) = (z_1z_2 + z_1z_3)$ Distributive law

5. $z + 0 = z$ The additive identity 0= (0, 0)

6. $z.1 = z$ Multiplicative identity 1= (1, 0)

7. $z = x + iy$ Then the additive inverse $-z = -(x + iy)$ such that $z + (-z) = 0$

8. $z = x + iy$ Then the multiplicative inverse

$$z^{-1} = \frac{1}{z} = \frac{1}{x+yi} = \frac{1}{x+yi}.\frac{x-yi}{x-yi}$$

$$\Longrightarrow \frac{x}{x^2 + y^2} - \frac{yi}{x^2 + y^2} = \left(\frac{x}{x^2 + y^2}, -\frac{yi}{x^2 + y^2}\right)$$

H .W. Show that

(a) $Re\,(iz) = -\,Im(z)$;

$(b)\ Im(iz) = Re(z)$.

Ex. $z_1 = (2,3)$ $z_2 = (4,5)$ find

1. $\frac{z_1}{z_2}$

2. z_1^{-1}

Sol.

1. $\frac{z_1}{z_2} = \frac{2+3i}{4+5i} \cdot \frac{4-5i}{4-5i} = \frac{23+2i}{16+25} = \frac{23}{41} + \frac{2i}{41}$.

يجب التخلص من (i)الموجودة في المقام

2. $z_1^{-1} = \frac{1}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{2-3i}{4+9} = \frac{2}{13} - \frac{3i}{13} \ni z_1 . z_1^{-1} = 1$

Verifying $(2 + 3i).(\frac{2}{13} - \frac{3i}{13}) = \frac{4}{13} - \frac{6i}{13} + \frac{6i}{13} - \frac{9i^2}{13} = \frac{4}{13} + \frac{9}{13} = \frac{13}{13} = 1$

Remark.

$$\sqrt{-a} = \sqrt{a}.\sqrt{-1} = \sqrt{a}.i$$

Ex: $\sqrt{-16} = \sqrt{16}.\sqrt{-1} = 4i$ & $\sqrt{-100} = 10i$ & $\sqrt{-13} = \sqrt{13}\, i$

Ex: Solve the equation $z^2 + 2 = 0$

Sol.

$$z^2 + 2 = 0 \to z^2 = -2 \to z = \pm\sqrt{-2i} \to \pm\sqrt{2}\, i$$

$$s = \{\sqrt{2}\, i, -\sqrt{2}\, i\}$$

Ex: Solve the equation $z^2 + 3zi - 2 = 0$

Sol. $z^2 + 3zi + 2i^2 = 0$

$$(z + 2i)(z + i) = 0$$

$$(z + 2i) = 0 \to z = -2i \quad or \quad (z + i) = 0 \to z = -i$$

$$s = \{-2i, -i\}$$

ملاحظة.

1. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$

Exercise:

1- Find the value of x, y that satisfy the eq.

$$(x - y - 6) + i\,(y^2 - x) = 0$$

2. Solve for real x, y the eq.

$$\left(\frac{1+i}{1-i}\right)^2 + \frac{1}{x+iy} = 1 + i$$

3. Solve the following equation $(3 - 2i)\,(x + iy) = 2(x - 2iy) + 2i - 1$

Moduli z المقياس

We define moduli $z = x + iy$ is $|z| = \sqrt{x^2 + y^2}$

Ex: find $|z|$ for 1.$z = 1 + 3i$ 2.$z = 1 - 3i$

Sol.

1. $|z| = \sqrt{x^2 + y^2} \to \sqrt{(1)^2 + (3)^2} = \sqrt{10}$

2. $|z| = \sqrt{x^2 + y^2} \to \sqrt{(1)^2 + (-3)^2} = \sqrt{10}$

Remark. For any complex number $z \to |z| \in \mathbb{R}$

$|3i|$=$|0 + 3i| = \sqrt{0 + 9} = 3$

$|i|$=1

Remark. $|z_1| < |z_2|$ Means that the point z_1 is closer to the origin than point z_2 if $z_1 < z_2$

الملاحظة أعلاه تدل ان حقل الاعداد المعقدة حقل غير مرتب والاقل أقرب الى نقطة الاصل

Now $|z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = |z_2 - z_1|$

أعلاه قانون المسافة بين نقطتي

Def: the complex **conjugate** or simply the conjugate of a complex number $z = x + iy$ is define as the complex number $\bar{z} = x - iy$

Properties

1. $z\bar{z} = |z|^2$ Very important $\Rightarrow$ $z\bar{z} = x^2 + y^2$

Ex.$(3,4).(3,-4) = 9 + 16 = 25$ & $|3 + 4i|^2 = \left(\sqrt{3^2 + 4^2}\right)^2 = \left(\sqrt{25}\right)^2 = 25$

حاصل ضرب عدد في مرافقه يكون الناتج عدد حقيقي

2. $|z_1 . z_2| = |z_1||z_2|$

Proof. From (1)

$|z_1 z_2|^2 = (z_1 . z_2).\overline{(z_1 . z_2)}$

$|z_1 z_2|^2 = (z_1 . z_2).(\overline{z_1} . \overline{z_2})$

$|z_1 z_2|^2 = (z_1 \overline{z_1})(z_2 \overline{z_2}) = |z_1|^2 |z_2|^2$ بالجذر

$$|z_1 . z_2| = |z_1||z_2|$$

2. $\bar{\bar{z}} = z \rightarrow z = x + iy$ Then $\bar{z} = x - iy \Rightarrow \bar{\bar{z}} = x + iy = z$

3. $|z| = |\bar{z}|$

Ex. $z = 2 + 5i \Longrightarrow \sqrt{(2)^2 + (5)^2} = \sqrt{29}$ and $\bar{z} = 2 - 5i \Longrightarrow \sqrt{(2)^2 + (-5)^2} = \sqrt{29}$

4. $z + \bar{z} = 2Rez \Longrightarrow \frac{z+\bar{z}}{2} = x$ Very important

5. $z - \bar{z} = 2iImz \Longrightarrow Im(z) = \frac{z-\bar{z}}{2i} = y$ Very important

6. $\frac{1}{z}=\frac{\bar{z}}{z\bar{z}}$

7. $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$

Proof.

$$\left|\frac{z_1}{z_2}\right|^2=\left(\frac{z_1}{z_2}\right).\overline{\left(\frac{z_1}{z_2}\right)}$$

$$\left|\frac{z_1}{z_2}\right|^2=\left(\frac{z_1}{z_2}\right).\left(\frac{\overline{z_1}}{\overline{z_2}}\right)$$

$$\left|\frac{z_1}{z_2}\right|^2=\frac{z_1\overline{z_1}}{z_2\overline{z_2}}=\frac{|z_1|^2}{|z_2|^2}\text{بالجذر}$$

$$\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$$

Prove that

8. $|z_1+z_2|\geq|z_1|-|z_2|$

9. $|z_1-z_2|\geq|z_1|-|z_2|$

10. $|z_1+z_2|\leq|z_1|+|z_2|$

11. $|z_1-z_2|\leq|z_1|+|z_2|$

Ex. Prove that $\left|(2\bar{z}+5)(\sqrt{2}-i)\right|=\sqrt{3}|2\bar{z}+5|$

Proof.

$$=|2\bar{z}+5|\left|\sqrt{2}-i\right|$$

$$=\sqrt{\left(\sqrt{2}\right)^2+(-1)^2}.|2\bar{z}+5|\rightarrow\sqrt{3}|2\bar{z}+5|$$

Ex.

If $iz^2-\bar{z}=0$ find values of $|z|$.

Sol. $iz^2 - \bar{z} = 0$

$iz^2 = \bar{z} \Longrightarrow |iz^2| = |\bar{z}|$ Where $|z| = |\bar{z}|$

$|i|.|z^2| = |z|$
$where$ $|\bar{z}|$=$|z|$and $|i| = 1$

$$|z^2| - |z| = 0$$

$$|z|[|z| - 1] = 0$$

Either $|z| = 0$ or $|z| - 1 = 0 \Longrightarrow |z| = 1$

Ex. If z lies on the circle $|z| = 2$ show that

1. $|z^2 + 2z - 1| \leq 9$

1. Sol. $|z^2 + 2z - 1| = |z^2 + 2z + (-1)| \leq |z|^2 + 2|z| + 1$

$$\leq 2^2 + 2(2) + 1$$

$$\leq 9$$

H.W. let $|z| = 1$ and $Re(z)^2 = 0$ find z

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5. Express the following equations in complex conjugate form.

عبر عن المعادلات الآتية بصيغة المرافق العقدي.

1) $2x + y = 5$, 2) $x^2 + y^2 = 36$

Q. Prove that the equation of the hyperbola is $x^2 - y^2 = 1$ is
$$z^2 + (\bar{z})^2 = 2$$

توضيح

اثبت أن :معادلة القطع الزائد $x^2 - y^2 = 1$ هي
$$z^2 + (\bar{z})^2 = 2$$