المحاضرة السادسة

دوال المتغير العقدي

Functions of complex variable.

Let S be a set of complex numbers. A function f defined on S is a rule that assigns to each $z \in S$ a single complex number W. In such case we write $W = f(z)$

ليكن S مجموعة من الأعداد المركبة. الدالة f المعرفة على S هي قاعدة تعين لكل $z \in S$ عنصر في المجال يرتبط بعنصر في المجال المقابل $w = f(z)$

Ex. Find the domain of the function

1.$f(z) = z^2 + z + 1$ (2).$f(z) = \frac{3}{z-3}$ (3).$f(z) = \frac{1}{z^2+1}$

Sol.

1) $\mathbb{C}$

2) $z - 3 = 0 \Longrightarrow z = 3 \quad \therefore \ \mathbb{C} - \{3\}$

3) $z^2 + 1 = 0 \Longrightarrow z^2 = -1 \Longrightarrow z = \pm\sqrt{-1} = \pm i \qquad \therefore \mathbb{C} - \{i, -i\}$

Real and imaginary parts of function

1. Let $w = f(z)$ and $z = x + yi$ be a function and let $w = u(x, y) + iv(x, y)$

Then $f(z) = f(x + yi) = u + iv$

2. if $z = re^{i\theta}$ then $f(z) = u(r, \theta) + iv(r, \theta)$ in polar coordinates

Ex. Write $f(z) = z^2$ in the forms

(1) $f(z) = u(x, y) + iv(x, y)$

(2) $f(z) = u(r, \theta) + iv(r, \theta)$

Sol. Let $z = x + yi \Longrightarrow f(z) = f(x + yi) = (x + yi)^2$

$x^2 - y^2 + 2xyi$

2. Let $z = re^{i\theta} \Longrightarrow f(z) = f\left(re^{i\theta}\right) = \left(re^{i\theta}\right)^2 = r^2 e^{2i\theta} = r^2[cos2\theta + isin2\theta]$

$$r^2 cos2\theta + r^2 isin2\theta$$

H.w. write the functions in the form $f(z) = u(x, y) + iv(x, y)$

(1) $f(z) = 2z^2 - 3z + i$ $\qquad$ $(2x^2 - 2y^2 - 2x + (4xy - 3y + 1)i$

(2) $f(z) = 2i\bar{z}$ $\qquad$ $(2y + 2xi)$

Ex. write the functions in the form. $f(z) = u(r, \theta) + iv(r, \theta)$

(1) $|z|$

Sol. let $z = re^{i\theta} \Longrightarrow$

$f(z) = \left|re^{i\theta}\right| \Longrightarrow f(z) = |r|.\left|e^{i\theta}\right| = r|cos\theta + isin\theta|$

$= r.\sqrt{cos\theta^2 + sin\theta^2} \Longrightarrow = r.1 = r$

The limited

Def. Suppose $\boldsymbol{f}$ is defined at all points in some neighborhood of a point z_0 by the statement that $\lim_{z \to z_0} f(z) = w_0$

$$\forall \in > \mathbf{0}, \exists \boldsymbol{\delta} > \mathbf{0} \ni |\boldsymbol{z} - \boldsymbol{z_0}| < \boldsymbol{\delta} \Longrightarrow |\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{w_0}| < \in$$

Ex. Let $f(z) = \frac{iz}{2}$ is defined on $|z| < 1$ prove that $\lim_{z\to 1} \frac{iz}{2} = \frac{i}{2}$ by definition

Sol.

$$\forall \in > \mathbf{0}, \exists \boldsymbol{\delta} > \mathbf{0} \ \ni |\boldsymbol{z} - \boldsymbol{z_0}| < \boldsymbol{\delta} \Longrightarrow |\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{w_0}| < \in$$

Clair that $z_0 = 1$, $f(z) = \frac{\boldsymbol{iz}}{\mathbf{2}}$ and $w_0 = \frac{\boldsymbol{i}}{\mathbf{2}}$

$\left|\frac{\boldsymbol{iz}}{\mathbf{2}} - \frac{\boldsymbol{i}}{\mathbf{2}}\right| < \in, |\boldsymbol{z} - \mathbf{1}| < \boldsymbol{\delta}$ …... (1)

$\frac{|\boldsymbol{i}|.|\boldsymbol{z}-\mathbf{1}|}{\mathbf{2}} < \in$ **But** $|\boldsymbol{i}| = \mathbf{1}$

$\Longrightarrow \frac{|\boldsymbol{z}-\mathbf{1}|}{\mathbf{2}} < \in \quad \Longrightarrow |\boldsymbol{z} - \mathbf{1}| < \mathbf{2} \in$ ….. (2)

From (1) & (2) $\boldsymbol{\delta} = \mathbf{2} \in$

Ex. prove that $\lim_{z\to 2i} z^2 = -4$ by definition

Sol.

Clair that $z_0 = 2i$, $f(z) = z^2$ and $w_0 = -4$

$$\forall \in > \mathbf{0}, \exists \boldsymbol{\delta} > \mathbf{0} \ \ni |\boldsymbol{z} - \boldsymbol{z_0}| < \boldsymbol{\delta} \Longrightarrow |\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{w_0}| < \in$$

$|z^2 + 4| < \in , |z - 2i| < \delta$ ….. (1)

$|z^2 - 4i^2| < \in \Longrightarrow |(z - 2i)(z + 2i)| < \in$ Where $z = 2i$

$\Longrightarrow |(z - 2i)||4i| < \in \ \Longrightarrow |(z - 2i)||4||i| < \in$ $\div 4$

$|(z - 2i)| < \frac{\in}{4}$ …… (2) From (1) & (2) $\delta = \frac{\in}{4}$

H.W prove that $\lim_{z\to -1} \frac{z^2+1}{z+i} = -2i$ by definition

Prove that $\lim_{z\to i} \frac{1}{z} = -i$ by definition

Continuity

Let $f(z)$ be a function defined in some neighborhood of the point z_0 then f is said to be continuous functions at z_0 IFF $\lim_{z\to z_0} f(z) = f(z_0)$ that is, $f(z)$ is continuous at z_0 IIF

$\forall \in > \mathbf{0}, \exists \boldsymbol{\delta} > \mathbf{0}\ \boldsymbol{suh\ that} |\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{z_0})| < \in\ \boldsymbol{when} |\boldsymbol{z} - \boldsymbol{z_0}| < \boldsymbol{\delta}$

Ex. Prove that $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{z} + \mathbf{1}$ is continuous at $\boldsymbol{z_0} = \mathbf{1}$ by definition

Sol.

$$\forall \in > \mathbf{0}, \exists \boldsymbol{\delta} > \mathbf{0}\ \boldsymbol{suh\ that} |\boldsymbol{f}(\boldsymbol{z}) - \boldsymbol{f}(\boldsymbol{z_0})| < \in\ \boldsymbol{when} |\boldsymbol{z} - \boldsymbol{z_0}| < \boldsymbol{\delta}$$

Clair that $f(z) = z + 1$, $z_0 = 1$ & $f(z_0) = f(1)$

$|z + 1 - f(1)| < \in$ when $|z - 1| < \delta$

$|z + 1 - 2| < \in$ when $|z - 1| < \delta$

$|z - 1| < \in$ when $|z - 1| < \delta$

$\in = \delta \Longrightarrow f$ is continuous

Ex. $f(z) = z^2$ is continuous at the point $z_0 = 3$

ملاحظة: -شرط الاستمرارية (الغاية = صورة الدالة) في حالة إيجاد استمرارية الدالة بدون تعريف

Sol.

$\lim_{z\to 3} z^2 = 9$ الغاية

$f(3^2) = 9$ الصورة

$\therefore \lim_{z\to 3} f(z) = f(3)\ f$ is continuous

Ex. Let $f(z) = \frac{z+i}{z-i}$ is continuous $z = 3i$

Sol. $z - i = 0 \rightarrow z = i$ يجب استبعاد النقاط التي تجعل المقام يساوي صفر

$$D = \mathbb{C} - \{i\}$$

$\lim_{z \to 3i} \frac{z+i}{z-i} = \frac{3i+i}{3i-i} = 2$ الغاية

$f(3i) = 2$ الصورة

$$\therefore \lim_{z \to 3i} f(z) = f(3i)$$

$$\therefore f \ continous$$

H.W. show that $f(z) = \frac{z^2 - iz + 4}{z}$ is continuous $z = i$