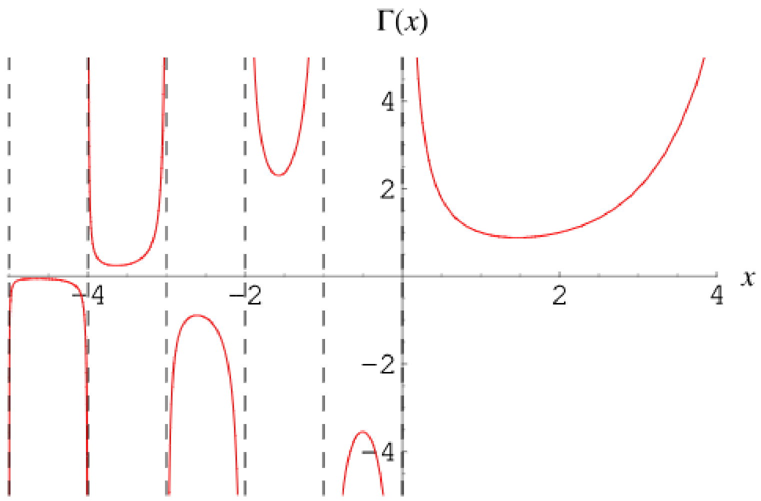
Gamma Function

Gamma function is defined by the integral

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \qquad ; n > 0$$



Plot of the Gamma Function

Gamma function satisfies the recursive properties:

1. $\Gamma(n+1) = n\Gamma(n) \qquad \forall\, n \neq 0 \;, n \notin \mathbb{Z}^-$
2. $\Gamma(n+1) = n! \qquad n \in \mathbb{N}$
3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
4. $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad ; \quad 0 < p < 1$

Example 1: Find 1. $\Gamma(5)$ 2. $\Gamma\left(\frac{3}{2}\right)$ 3. $\Gamma\left(\frac{5}{2}\right)$

4. $\Gamma\left(-\frac{1}{2}\right)$ 5. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

1. $\Gamma(5) = 4! = 4 \times 3 \times 2 \times 1 = 24$
2. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}+1\right) = \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\sqrt{\pi}$

3. $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}$

4. $\Gamma(n+1) = n\Gamma(n) \quad \Rightarrow \quad \Gamma(n) = \frac{\Gamma(n+1)}{n}$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\,\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

5. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\,\pi$

Example 2: Evaluate each of the following integrals

1. $\int_0^\infty x^5\, e^{-x} dx = \Gamma(5+1) = 5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

2. $\int_0^\infty x\sqrt{x}\; e^{-x} dx = \int_0^\infty x^{\frac{3}{2}}\, e^{-x} dx \; = \Gamma\left(\frac{3}{2}+1\right)$

$$= \frac{3}{2} \times \frac{1}{2}\sqrt{\pi} = \frac{3}{4}\sqrt{\pi}$$

3. $\int_0^\infty x^4\, e^{-2x} dx$

Let $u = 2x \quad \Rightarrow \quad du = 2dx \quad \Rightarrow \quad dx = \frac{du}{2}$

$$\int_0^\infty x^4\, e^{-2x} dx = \int_0^\infty \left(\frac{u}{2}\right)^4 e^{-u}\frac{du}{2} = \frac{1}{2^5}\int_0^\infty u^4\, e^{-u} dx$$

$$= \frac{1}{32}\Gamma(5) = \frac{4!}{32}$$

$$= \frac{24}{32} = \frac{3}{4}$$

4. $$\int_0^\infty \frac{dx}{\sqrt[3]{x^2}(1+x)} = \int_0^\infty \frac{x^{-2/3}}{1+x}dx$$

$$= \int_0^\infty \frac{x^{(1/3)-1}}{1+x}dx = \frac{\pi}{\sin\frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

Beta Function

Beta function is defined by the integral

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx; \quad n > 0, m > 0$$

Beta function satisfies the recursive properties:

1. The Beta function is symmetric that is : $B(m,n) = B(n,m)$

2. $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

3. $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta = \frac{1}{2}B(m,n)$

Example 3: Evaluate 1. $B(3,4)$ 2. $B\left(\frac{1}{2},\frac{5}{2}\right)$

1. $$B(3,4) = \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2!\times 3!}{6!}$$

$$= \frac{2\times 3!}{6\times 5\times 4\times 3!} = \frac{1}{60}$$

2. $$B\left(\frac{1}{2},\frac{5}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{5}{2}\right)}$$

$$= \frac{\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{\Gamma(3)} = \frac{3\pi}{8}$$

Example 4: Evaluate each of the following integrals

1. $$\int_0^1 x^3(1-x)^4\,dx = B(4,5) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3!\times 4!}{8!}$$

$$= \frac{6\times 4!}{8\times 7\times 6\times 5\times 4!} = \frac{1}{280}$$

2. $$\int_0^{\pi/2} \sin^9\theta\cos^5\theta\,d\theta$$

$$2m-1=9 \;\Rightarrow\; m=5 \quad \text{and} \quad 2n-1=5 \;\Rightarrow\; n=3$$

$$\int_0^{\pi/2} \sin^9\theta\cos^5\theta\,d\theta = \frac{1}{2}B(5,3) = \frac{\Gamma(5)\Gamma(3)}{2\Gamma(5+3)} = \frac{4!\times 2!}{2\times 7!} = \frac{1}{210}$$

3. $$\int_0^{\pi/2} \sin^5 x\,dx \;;\; 2m-1=5 \;\Rightarrow\; m=3 \quad \text{and} \quad 2n-1=0 \;\Rightarrow\; n=\frac{1}{2}$$

$$\int_0^{\pi/2} \sin^5 x\,dx = \frac{1}{2}B\left(3,\frac{1}{2}\right) = \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)} = \frac{2!\sqrt{\pi}}{2\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{8}{15}$$

Exercises

Evaluate each of the following integrals

1. $\displaystyle\int_0^\infty x^5 e^{-3x}dx$

2. $\displaystyle\int_0^\infty \frac{1}{\sqrt[4]{x}\,(1+x)}dx$

3. $\displaystyle\int_0^1 x^5(1-x)^6\,dx$

4. $\displaystyle\int_0^{\pi/2} \cos^4 x\,dx$

5. $\displaystyle\int_0^{\pi/2} \sin^5 x\cos^4 x\,dx$