Second -Order ODE

1-Homogeneous Linear ODE with Constant Coefficients

A second order homogeneous equation with constant coefficients is written as

$ay'' + by' + cy = 0$ where a, b and c are constant.

Let us summarize the steps to follow in order to find the general solution:

1. Write down the **characteristic equation** $am^2 + bm + c = 0$
2. Find the roots of the characteristic equation m_1 and m_2. Here we have three cases:

$\Delta = b^2 - 4ac$	m_1 and m_2	General solution y_h
$\Delta > 0$	$m_1 \neq m_2$	$y_h = c_1e^{m_1x} + c_2e^{m_2x}$
$\Delta = 0$	$m_1 = m_2$	$y_h = (c_1x + c_2)e^{mx}$
$\Delta < 0$	$m_1 = \overline{m_2} = \alpha + \beta i$	$y_h = e^{\alpha x}(c_1 \sin \beta x + c_2 \cos \beta x)$

Example 1: Find the general solution of ODE $y'' - 6y' + 8y = 0.$

Solution: Characteristic equation and its roots:

$$m^2 - 6m + 8 = 0 \Rightarrow (m - 2)(m - 4) = 0 \Rightarrow m_1 = 2 \text{ and } m_2 = 4$$

The general solution is: $y_h = c_1e^{2x} + c_2e^{4x}$

Example 2: Find the general solution of ODE $y'' - y' - 20y = 0.$

Solution: $m^2 - m - 20 = 0$ (Characteristic equation)

$(m + 4)(m - 5) = 0 \Rightarrow m_1 =- 4$ and $m_2 = 5$ (The roots)

$y_h = c_1e^{-4x} + c_2e^{5x}$ (General solution)

Example 3: Find the general solution of ODE $y'' - 6y' + 9y = 0.$

Solution: $m^2 - 6m + 9 = 0 \Rightarrow (m - 3)(m - 3) = 0 \Rightarrow m_1 = m_2 = 3$

$y_h = (c_1x + c_2)e^{3x}$

Example 4: Find the general solution of ODE $y'' - 2y' + 5y = 0.$

Solution: $m^2 - 2m + 5 = 0$ (Characteristic equation)

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-(-2) \mp \sqrt{(-2)^2 - 4 \times 1 \times 5}}{2 \times 1} = 1 \mp 2i$$

$y_h = e^x(c_1 \sin 2x + c_2 \cos 2x)$

Example 5: Solve $y'' + 2y' + 10y = 0$ with $y'(0) = 9$ and $y(0) = 3$

Solution $m^2 + 2m + 10 = 0$

$$m = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \mp \sqrt{(2)^2 - 4 \times 1 \times 10}}{2 \times 1} = -1 \mp 3i$$

$y_h = e^{-x}[c_1 \sin 3x + c_2 \cos 3x]$ (General solution)

$y(0) = 3 \Rightarrow 3 = e^0[c_1 \sin 0 + c_2 \cos 0] \Rightarrow c_2 = 3$

$y_h = e^{-x}[c_1 \sin 3x + 3 \cos 3x]$

$y_h' = e^{-x}[3c_1 \cos 3x - 9 \sin 3x] - e^{-x}[c_1 \sin 3x + 3 \cos 3x]$

$y'(0) = 9 \Rightarrow 9 = 3c_1 - 3 \Rightarrow c_1 = 4$

$y_h = e^{-x}[4 \sin 3x + 3 \cos 3x]$ (Particular solution)

Example 6: Solve $y'' + 9y = 0$ with $y'(\pi/6) = 3$ and $y(\pi/6) = 2$

Solution $m^2 + 9 = 0 \Rightarrow m = \mp 3i$

$y_h = c_1 \sin 3x + c_2 \cos 3x$ (General solution)

$y(\pi/6) = 2 \Rightarrow 2 = c_1 \sin(\pi/2) + c_2 \cos(\pi/2) \Rightarrow c_1 = 2$

$y'_h = 3c_1 \cos 3x - 3c_2 \sin 3x$

$y'(\pi/6) = 3 \Rightarrow c_2 = -1$

$y_h = 2 \sin 3x - \cos 3x$ (Particular solution)

Example 7: Solve $y'' + 9y' + 14y = 0$ with $y'(5\pi) = 2$ and $y(5\pi) = 4$

Solution: $m^2 + 9m + 14 = 0 \Rightarrow (m+2)(m+7) = 0 \Rightarrow m = -2, -7$

$y_h = c_1 e^{-2x} + c_2 e^{-7x}$ (General solution)

$y(5\pi) = 4 \Rightarrow 4 = c_1 e^{-10\pi} + c_2 e^{-35\pi} \quad \cdots \ eq(1)$

$y_h' = -2c_1 e^{-2x} - 7c_2 e^{-7x}$

$y'(5\pi) = 2 \Rightarrow 2 = -2c_1 e^{-10\pi} - 7c_2 e^{-35\pi} \quad \cdots \ eq(2)$

$eq(1) \times 2 \Rightarrow 8 = 2c_1 e^{-10\pi} + 2c_2 e^{-35\pi}$ by summation.

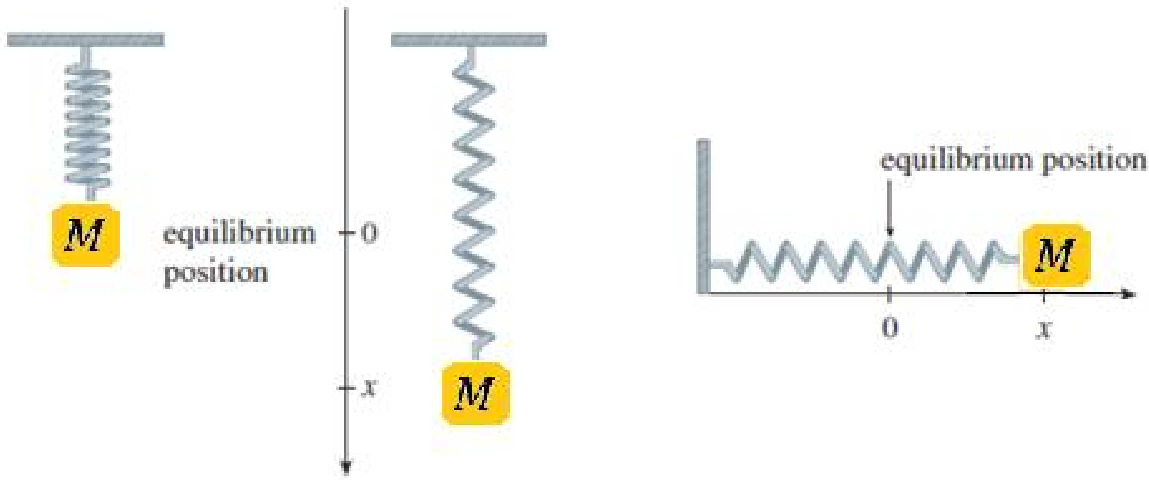
We get: $10 = -5c_2 e^{-35\pi} \Rightarrow \boxed{c_2 = -2e^{35\pi}}$ and $\boxed{c_1 = 6e^{10\pi}}$

$y_h = 6e^{10\pi} e^{-2x} - 2e^{35\pi} e^{-7x}$

$y_h = 6e^{10\pi - 2x} - 2e^{35\pi - 7x}$ (Particular solution)

Simple Harmonic Motion

Simple Harmonic Motion is defined as a motion in which the restoring force is directly proportional to the displacement of the body from its position. We consider the motion of an object with mass M at the end of a spring that is either vertical or horizontal on a level surface as in a Figure.



Hooke's Law, says that if the spring is stretched units from its natural length, then it exerts a force that is proportional to x.

Restoring Force $F =- kx$, where $k > 0$ is the spring constant.

If we ignore any external resisting forces then, by Newton's Second Law of motion we have $F = Ma$.

But $a = \dfrac{d^2x}{dt^2}$ ⇨ $F = M\dfrac{d^2x}{dt^2}$

$$M\frac{d^2x}{dt^2} =- kx$$

$$\boxed{M\frac{d^2x}{dt^2} + kx = 0 \text{ or } Mx'' + kx = 0};\ x'' = \frac{d^2x}{dt^2}$$

Example 8: A frictionless spring with a $10kg$ mass can be held stretched $1\ m$ beyond its natural length by a force of $40\ N$. If the spring begins at its equilibrium position, but a push gives it an initial velocity of $2.5\ m/sec$, find the position of the mass after t seconds.

Solution: From Hooke's Law, the force required to stretch the spring is:

$$40 = 1k \Rightarrow k = 40\ N/m$$

$$10x'' + 40x = 0$$

The IVP becomes:

$$x'' + 4x = 0 \text{ with } x(0) = 1 \text{ and } x'(0) = 2.5$$

$$m^2 + 4 = 0 \Rightarrow m = \mp 2i$$

$$x = c_1 \sin 2t + c_2 \cos 2t$$

$$x(0) = 1 \Rightarrow c_2 = 1$$

$$x' = 2c_1 \cos 2t - 2c_2 \sin 2t$$

$$x'(0) = 2.5 \Rightarrow 2.5 = 2c_1 \Rightarrow c_1 = 1.25$$

$$x = 1.25 \sin 2t + \cos 2t$$

Example 9: A spring with a mass of $2\ kg$ has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7m$. If the spring is stretched to a length of $0.7\ m$ and then released with initial velocity 0, find the position of the mass at any time.

Solution: Stretch on length $= 0.7 - 0.5 = 0.2\ m$

$$25.6 = 0.2k \Rightarrow k = 128\ N/m$$

$$2x'' + 128x = 0 \Rightarrow x'' + 64x = 0 \text{ with } x(0) = 0.2 \text{ and } x'(0) = 0$$

$$m^2 + 64 = 0 \Rightarrow m = \mp 8i$$

$$x = c_1 \sin 8t + c_2 \cos 8t$$

$$x(0) = 0.2 \Rightarrow c_2 = 0.2$$

$$x' = 8c_1 \cos 8t - 8c_2 \sin 8t$$

$$x'(0) = 0 \Rightarrow 0 = 8c_1 \Rightarrow c_1 = 0$$

$$x = 0.2 \cos 8t$$

Damped Vibrations

We next consider the motion of a spring that is subject to a frictional force. An example is the damping force supplied by a shock absorber in a car or a bicycle. We assume that the damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion. Then

damping force $= -c\dfrac{dx}{dt}$, where $c > 0$ is the damping constant.

Thus, in this case, Newton's Second Law gives:

$$M\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0.$$

Example 10: A spring with a mass of $2\ kg$ has natural length $0.5\ m$. A force of $25.6\ N$ is required to maintain it stretched to a length of $0.7\ m$. If the spring is immersed in a fluid with damping constant $c = 40$. Find the position of the mass at any time if it starts from the equilibrium position and is given a push to start it with an initial velocity of $0.6\ m/s$.

Solution: $25.6 = 0.2k \;\Rightarrow\; k = 128\,,\ M = 2$ and $c = 40$

$$2\frac{d^2x}{dt^2} + 40\frac{dx}{dt} + 128x = 0$$

$$2x'' + 40x' + 128x = 0 \;\; ; \;\; x(0) = 0 \text{ and } x'(0) = 0.6$$

$$x'' + 20x' + 64x = 0$$

$$m^2 + 20m + 64 = 0 \;\Rightarrow (m+4)(m+16) = 0 \;\Rightarrow\; m = -4\,, -16$$

$$x(t) = c_1e^{-4t} + c_2e^{-16t}$$

$$x(0) = 0 \quad \Rightarrow c_1 + c_2 = 0 \;\cdots(1)$$

$$x'(t) = -4c_1e^{-4t} - 16c_2e^{-16t}$$

$$-4c_1 - 16c_2 = 0.6 \qquad \cdots(2)$$

$$c_1 = 0.05 \text{ and } c_2 = -0.05$$

$$x(t) = 0.05\,\left(e^{-4t} - e^{-16t}\right)$$

Exercises

For each of the following problem (1 through 10), find the general solution of ODEs

(1) $y'' + y' - 2y = 0$ (2) $y'' - 4y' + 4y = 0$

(3) $y'' + 2y' + 2y = 0$ (4) $y'' + 2y' = 0$

(5) $y'' + 6y' + 5y = 0$ (6) $y'' + 2y' + 4y = 0$

(7) $y'' + 9y' + 8y = 0$ (8) $y'' - 6y' + 25y = 0$

(9) $y'' - 6y' + 9y = 0$ (10) $y'' - 4y' + 13y = 0$

For each of the following problem (11 through 14), solve IVPs

(11) $y'' + 5y' + 4y = 0$; $y'(0) =- 7$ and $y(0) = 1$

(12) $y'' + y' - 12y = 0$; $y'(\pi) =- 20$ and $y(\pi) =- 2$

(13) $9y'' + y = 0$; $y'(0) = 2$ and $y(0) =- 2$

(14) $y'' + 2y' - 4y = 0$; $y'(0) =- 6$ and $y(0) = 6$

(15) A spring with a 3 kg mass is held stretched 0.6 m beyond its natural length by a force of 20 N. If the spring begins at its equilibrium position but a push gives it an initial velocity of 1.2 m/s, find the position of the mass after t seconds.

(16) A spring with a mass of $3\ kg$ has damping constant $c = 30$ and spring constant $k = 123$. Find the position of the mass at time t if it starts at the equilibrium position with a velocity of $2\ m/s$.