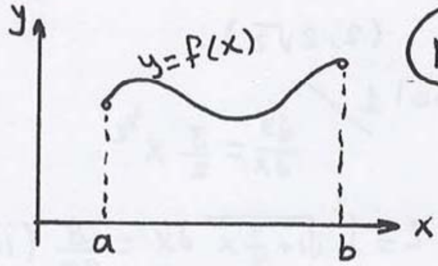
* Length of a plane Curve



- If $y=f(x)$ is a smooth Curve on the interval $[a,b]$, Then The arc length L of This Curve over $[a,b]$ is

$$L=\int_a^b \sqrt{1+[f'(x)]^2}\,dx=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

- but, for a curve expressed in The form $x=g(y)$ on The interval $[c,d]$, The arc length L is

$$L=\int_c^d \sqrt{1+[g'(y)]^2}\,dy=\int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

- If no segment of The curve represented by The parametric equations $x=x(t)$, $y=y(t)$ $(a\leq t\leq b)$ Then The arc length L of The Curve is

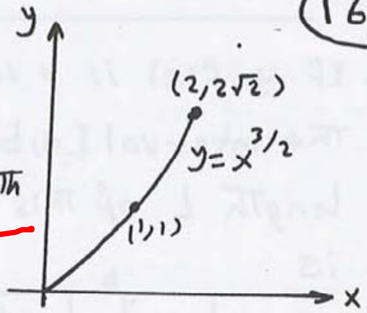
$$L=\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

* length of arc of The polar curve $r=f(\theta)$ between $\theta=\theta_1$ to θ_2

$$L=\int_{\theta_1}^{\theta_2} \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$$

See fig a page 151

Ex/ Find the arc length of the curve $y=x^{3/2}$ from $(1,1)$ to $(2,2\sqrt{2})$.



Sol 1/

$$\frac{dy}{dx}=\frac{3}{2}x^{1/2}$$

$$L=\int_1^2\sqrt{1+\frac{9}{4}x}\;dx=\frac{8}{27}\left(1+\frac{9}{4}x\right)\Big]_1^2\simeq 2.0856 \text{ u. length}$$

Sol 2/

$$\frac{4}{9}\times\frac{2}{3}$$

$$\frac{dx}{dy}=\frac{2}{3}y^{-\frac{1}{3}}$$

$$L=\int_1^{2\sqrt{2}}\sqrt{1+\frac{4}{9}y^{-2/3}}\;dy=\int_1^{2\sqrt{2}}y^{-\frac{1}{3}}\sqrt{y^{2/3}+\frac{4}{9}}\;dy=\left(y^{\frac{2}{3}}+\frac{4}{9}\right)^{\frac{3}{2}}\Big]_1^{2\sqrt{2}}$$

$$\simeq 2.0856 \text{ u. length}$$

Ex/ Find the circumference of a circle of radius a from the parametric equations

$x=a\cos t,\quad y=a\sin t \qquad 0\leq t\leq 2\pi$

Sol/ $\dfrac{dx}{dt}=-a\sin t \quad , \quad \dfrac{dy}{dt}=a\cos t$

$$L=\int_0^{2\pi}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt=\int_0^{2\pi}\sqrt{(-a\sin t)^2+(a\cos t)^2}\,dt$$

$$=\int_0^{2\pi}a\,dt=at\Big]_0^{2\pi}=2\pi a \quad \text{u. length}$$

Ex/ Find the length of the arc of spiral $r=ae^{3\theta}$ from $\theta=0$ to 2π

Sol/ $\dfrac{dr}{d\theta}=3ae^{3\theta} \rightarrow \quad r^2=1a^2\left(e^{3\theta}\right)^2 \;;\; \left(\dfrac{dr}{d\theta}\right)^2=9a^2\left(e^{3\theta}\right)^2$

$$L=\int_0^{2\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\frac{1}{3}\int 3\sqrt{10a^2(e^{3\theta})^2}\,d\theta=\frac{\sqrt{10a^2}}{3}\left[e^{3\theta}\right]_0^{2\pi}$$

$$=\frac{a\sqrt{10}}{3}\left[e^{6\pi}-1\right] \quad \text{u. length}$$

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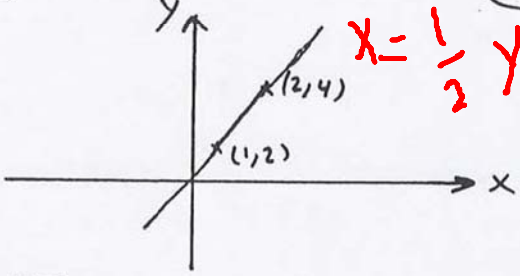
Ex/ Find The Length of The line segment $y=2x$ from $(1,2)$ to $(2,4)$.

$x = \frac{1}{2} y$



Sol/

way 1/ $\frac{dy}{dx} = 2$

$$L = \int_1^2 \sqrt{1+4}\, dx = \sqrt{5} \text{ u. length}$$

way 2/ $x = \frac{1}{2} y \rightarrow \frac{dx}{dy} = \frac{1}{2}$

$$L = \int_2^4 \sqrt{1+\frac{1}{4}}\, dy = \frac{\sqrt{5}}{2}[4-2] = \sqrt{5} \text{ u. length}$$

way 3/ $L = \sqrt{(y_2-y_1)^2 + (x_2-x_1)^2} = \sqrt{(4-2)^2+(2-1)^2} = \sqrt{5}$ u. length

HW/ Find The exact arc length of The Curve if

ⓐ $y = 3x^{3/2} - 1$ from $x=0$ to $x=1$ Ans/ $(85\sqrt{85}-8)/243$

ⓑ $y = x^{2/3}$ from $x=1$ to $x=8$ Ans/ $\frac{1}{27}(80\sqrt{10} - 13\sqrt{13})$

ⓒ $24xy = y^4 + 48$ from $y=2$ to $y=4$ Ans/ $17/6$

ⓓ $x = \frac{1}{3}t^3$, $y = \frac{1}{2}t^2$ $0 \le t \le 1$ Ans/ $(2\sqrt{2}-1)/3$

ⓔ $x = \cos 2t$, $y = \sin 2t$ $0 \le t \le \pi/2$ Ans/ π

ⓕ Cardioid $r = a(1+\cos\theta)$ between $\theta = 0$ to π Ans/ $4a$

* Area of a surface of revolution

$$S=\int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\,dx$$

$$=\int_a^b 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

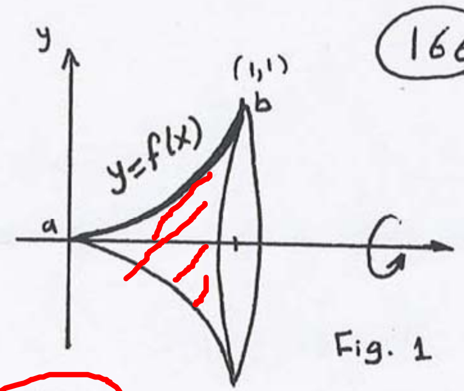


Fig. 1

$$S=\int_c^d 2\pi g(y)\sqrt{1+[g'(y)]^2}\,dy$$

$$=\int_c^d 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy$$

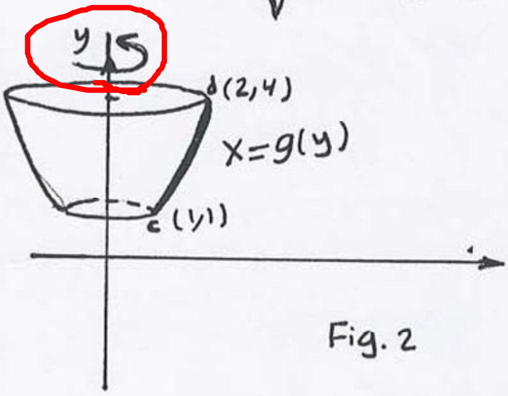


Fig. 2

Ex/ Find the area of the surface that is generated by revolving the portion of the curve with interval and revolving shown below:

ⓐ $y=x^3$ between $x=0$ and $x=1$ about x-axis.
ⓑ $y=x^2$ between $x=1$ and $x=2$ about y-axis.

sol/ ⓐ $\frac{dy}{dx}=3x^2$

$$S=\int_0^1 2\pi y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx=\int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx$$

$$\frac{1}{18}\times\frac{2}{3}=\frac{1}{27}$$

$$=\frac{\cancel{2}\pi}{\cancel{36}_{18}}\;\frac{(1+9x^4)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg]_0^1=\frac{\pi}{27}\left[(10)^{\frac{3}{2}}-1\right]=1.134\pi\simeq 3.563\text{ u.area}$$

See Fig.1

ⓑ $x=y^{\frac{1}{2}}\longrightarrow\frac{dx}{dy}=\frac{1}{2}y^{-\frac{1}{2}}=\frac{1}{2\sqrt{y}}$

$$S=\int_1^4 2\pi x\sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy=\int_1^4 2\pi\sqrt{y}\sqrt{1+\left(\frac{1}{2\sqrt{y}}\right)^2}\,dy$$

$$=2\pi\int_1^4\left(y+\frac{1}{4}\right)^{1/2}dy=\frac{4\pi}{3}\left[\left(y+\frac{1}{4}\right)^{3/2}\right]_1^4$$

$$=\frac{4\pi}{3}\left[\left(\frac{17}{4}\right)^{\frac{3}{2}}-\left(\frac{5}{4}\right)^{3/2}\right]=\frac{4\pi}{3}[7.364..]\simeq 30.8446\text{ u.area}$$

see Fig.2

HW/ Find The area of The surface generated by revolving The given curve about The indicated axis (167)

ⓐ $y = 7x$ $0 \leq x \leq 1$ about x-axis   Ans/ $35\sqrt{2}\,\pi$

ⓑ $y = \sqrt{4 - x^2}$ $-1 \leq x \leq 1$ about x-axis   Ans/ 8π

ⓒ $x = 9y + 1$ $0 \leq y \leq 2$ about y-axis   Ans/ $40\sqrt{82}\,\pi$

ⓓ $x = \sqrt{9 - y^2}$ $-2 \leq y \leq 2$ about y-axis   Ans/ 24π

* Area of The surface in polar Co-ordinates system.

$$S = \int_{\theta_1}^{\theta_2} 2\pi r \cdot \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

see fig a page 96

EX/ find The surface area generated when The arc of The Curve $r = 5(1 + \cos\theta)$ between $\theta = 0$ to π rotate about The intial line

sol/

$$S = \int_0^{\pi} 2\pi \cdot 5(1+\cos\theta) \cdot \sin\theta \sqrt{25(1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}\, d\theta$$

$$= 10\pi \int_0^{\pi} (1+\cos\theta) \cdot \sin\theta \sqrt{50\left(1 + 2\cos^2\frac{\theta}{2} - 1\right)}$$

$$= 10\pi \int_0^{\pi} (1+\cos\theta) \cdot \sin\theta \cdot 10 \cdot \cos\frac{\theta}{2}\, d\theta$$

$$= 100\pi \int_0^{\pi} 2\cos^2\frac{\theta}{2} \cdot 2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}\, d\theta$$

$$= 400\pi \int_0^{\pi} \cos^4\frac{\theta}{2} \cdot \sin\frac{\theta}{2}\, d\theta$$

$$= -800\pi \left[\frac{\cos^5\frac{\theta}{2}}{5}\right]_0^{\pi} = 160\pi \text{ sq.u.}$$

HW/ Find The area of The surface generated when The arc of The curve $r = ae^{\theta}$ between $\theta = 0$ to $\frac{\pi}{2}$ · rotate about intial line

Ans/ $\frac{2\sqrt{2}}{5}\pi a^2 (2e^{\pi} + 1)$