Chapter One
Transient Analysis of RL, RC, and RLC Circuits

Transient Analysis of RL, RC, and RLC Circuits will be divided into two parts, the first one is the natural response and the second one is the step response.

The natural response of a circuit refers to the behavior (in terms of voltages and currents) of the circuit itself, with no external sources of excitation.

The step response of a circuit is its behavior when the excitation is the step function, which may be a voltage or a current source.

1.1 Natural Response of RL Circuits

The following circuit in figure 1.1 is an example of an RL circuit. The switch is closed for a long time and the inductor appears as a short circuit. The goal is to find $v(t)$ and $i(t)$ after the switch is opened ($t \geq 0$).

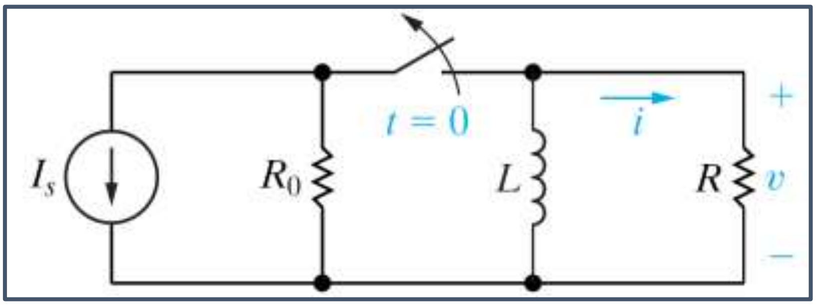

Figure 1.1

At time $t \geq 0$, the current $i(t)$ can be find using Kirchhoff's voltage law to obtain an expression involving $i, R, and\ L$.

$$L\frac{di}{dt} + Ri = 0$$ First order differential equation

$$\frac{di}{dt}dt = -\frac{R}{L}idt$$ Multiply both sides by a differential time dt

$$\frac{di}{i} = -\frac{R}{L}dt$$ Divide through by i

Integrating both sides of the above equation using x and y as variables of integration yields:

$$\int_{i(t_o)}^{i(t)} \frac{dx}{x} = -\frac{R}{L}\int_{t_o}^{t} dy$$

$i(t_o)$; is the current corresponding to the time t_o and $i(t)$ is the current corresponding to time t. Here $t_o = 0$, therefore, carrying out the indicated integration gives:

$ln\frac{i(t)}{i(0)} = -\frac{R}{L}t$, $i(0)$: is the initial condition

$$i(t) = i(0)e^{-(R/L)t}$$

This shows that the natural response of the RL circuit is an exponential decay of the initial current. At time t = 0 the inductor has an initial current $i(0) = I_0$, then:

$i(t) = I_o e^{-(R/L)t}, \ t \geq 0$ (**Natural response of an *RL* circuit**)

The above equation shows that the current starts from an initial value I_0 and decreases exponentially toward zero as t increases as shown in figure 1.2.

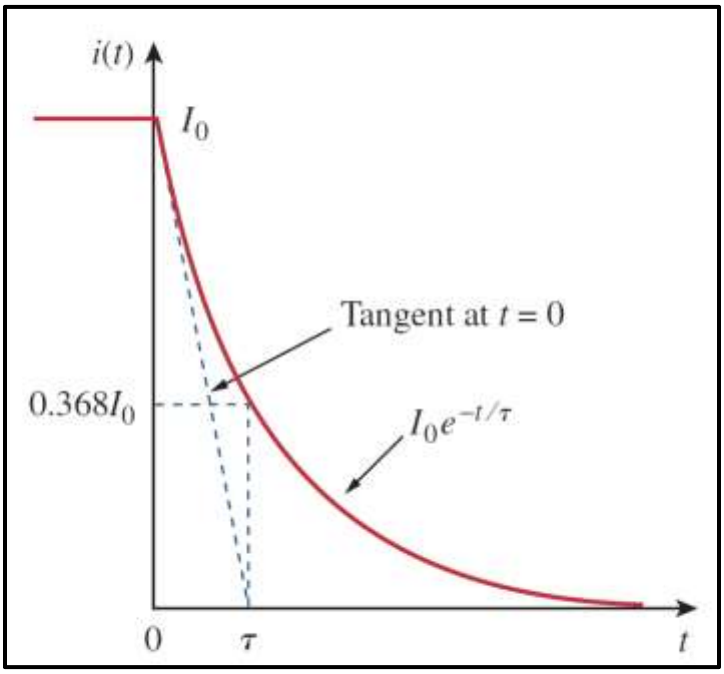

Figure 1.2

The coefficient of t, namely, R/L determines the rate at which the current or voltage approaches zero. The reciprocal of this ratio is the ***time constant*** of the circuit:

The time constant of a circuit is the time required for the response to decay to a factor of 1/e or 36.8 percent of its initial value.

$$\tau = \text{timeconstant} = \frac{L}{R}$$

Calculating the natural response of an RL circuit can be summarized as follows:

1. ***Find the initial current I_0, through the inductor.***

2. Find the time constant of the circuit $\tau = \frac{L}{R}$.

3. Use the equation $\left[i(t) = I_o e^{-(R/L)t}\right]$ to generate $i(t)$ from I_0 and τ.

1.2 Natural Response of RC Circuits

The natural response of an RC circuit is developed from the circuit shown in figure 1.3. The switch has been closed for a long time (t < 0). Recall that a capacitor behaves as an open circuit in the presence of a constant voltage.

At time t > 0, the switch is moved to position b and the voltage source and the resistance R1 are disconnected.

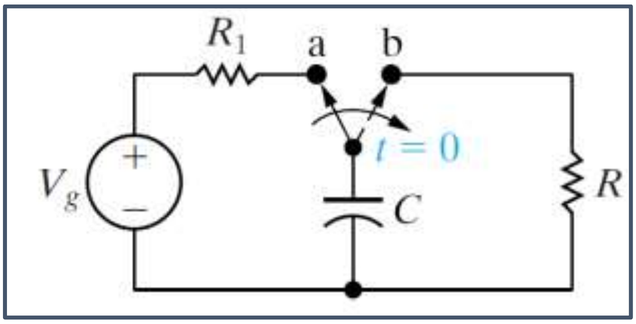

Figure 1.3

Summing the currents in the resultant circuit:

$$C\frac{dv}{dt} + \frac{v}{R} = 0.$$

The same mathematical techniques can be used to obtain the solution for $v(t)$

$$v(t) = v(0)e^{-t/RC}, \quad t \geq 0.$$

The initial capacitor voltage is $v(0) = V_0$ (the voltage source voltage)

The time constant for the RC circuit is $\boldsymbol{\tau = RC}$

$v(t) = V_0 e^{-t/\tau}, t \geq 0,$ **(The natural response of an RC circuit)**

This indicates that the natural response of an *RC* circuit is an exponential decay of the initial voltage, governed by the time constant *RC* as shown in figure 1.4

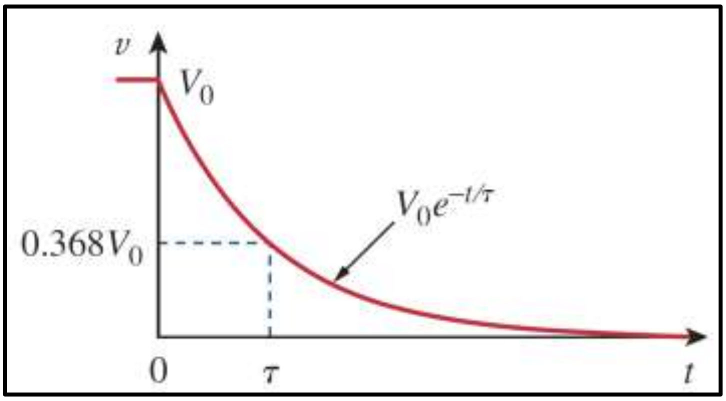

Figure 1.4

The expressions for $i, p, and\ \omega$ are:

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R}e^{-t/\tau}, \quad t \geq 0^+,$$

$$p = vi = \frac{V_0^2}{R}e^{-2t/\tau}, \quad t \geq 0^+,$$

$$w = \int_0^t p\,dx = \int_0^t \frac{V_0^2}{R}e^{-2x/\tau}\,dx$$

$$= \frac{1}{2}CV_0^2(1 - e^{-2t/\tau}), \quad t \geq 0.$$

Calculating the natural response of an RC circuit can be summarized as follows:

1. ***Find the initial voltage V_0, across the capacitor.***

2. Find the time constant of the circuit $\tau = RC$

3. Use the equation $\left[v(t) = V_0e^{-\frac{t}{\tau}}\right]$ to generate $v(t)$ from V_0 and τ

1.3 The Step Response of an RL Circuit

Consider the circuit in figure 1.5, the switch is opened for a long time, the goal is to find an expression for the current in the circuit and the voltage across the inductor.

At time t = 0 when the switch is closed using Kirchhoff's voltage law

$$V_s = Ri + L\frac{di}{dt},$$

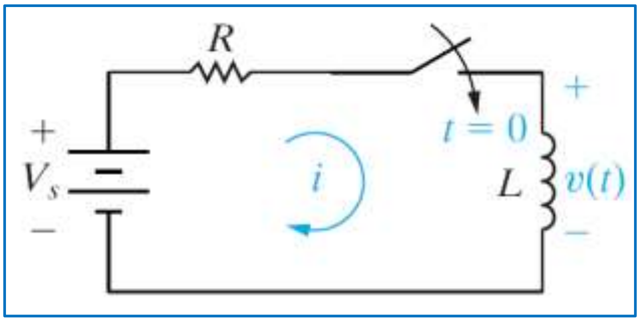

Figure 1.5

The first step in this approach is to solve the above equation for the derivative di/dt;

$$\frac{di}{dt} = \frac{-Ri + V_s}{L} = \frac{-R}{L}\left(i - \frac{V_s}{R}\right).$$

Multiply both sides of the above equation by a differential time dt results in:

$$\frac{di}{dt}dt = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)dt, \qquad di = \frac{-R}{L}\left(i - \frac{V_s}{R}\right)dt.$$

$$\frac{di}{i - (V_s/R)} = \frac{-R}{L}dt,$$

Using x and y as variables for the integration, we obtain

$$\int_{I_0}^{i(t)} \frac{dx}{x - (V_s/R)} = \frac{-R}{L}\int_0^t dy,$$

Where I_0 is the current at $t = 0$ and $i(t)$ is the current at any $t > 0$

$$\ln\frac{i(t) - (V_s/R)}{I_0 - (V_s/R)} = \frac{-R}{L}t, \quad \frac{i(t) - (V_s/R)}{I_0 - (V_s/R)} = e^{-(R/L)t},$$

The step response of an RL circuit is:

$$i(t) = \frac{V_s}{R} + \left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t}.$$

When the initial energy in the inductor is zero, I_0 is zero:

$$i(t) = \frac{V_s}{R} - \frac{V_s}{R}e^{-(R/L)t}.$$

One time constant after the switch has been closed, the current will have reached approximately 63% of its final value (figure 1.6), or

$$i(\tau) = \frac{V_s}{R} - \frac{V_s}{R}e^{-1} \approx 0.6321\frac{V_s}{R}.$$

The voltage across an inductor is Ldi/dt so:

$$v = L\left(\frac{-R}{L}\right)\left(I_0 - \frac{V_s}{R}\right)e^{-(R/L)t} = (V_s - I_0R)e^{-(R/L)t}.$$

The voltage across the inductor is zero before the switch is closed. The above Equation indicates that the inductor voltage jumps to $V_s - I_0R$ at the instant the switch is closed and then decays exponentially to zero as shown in figure 1.7.

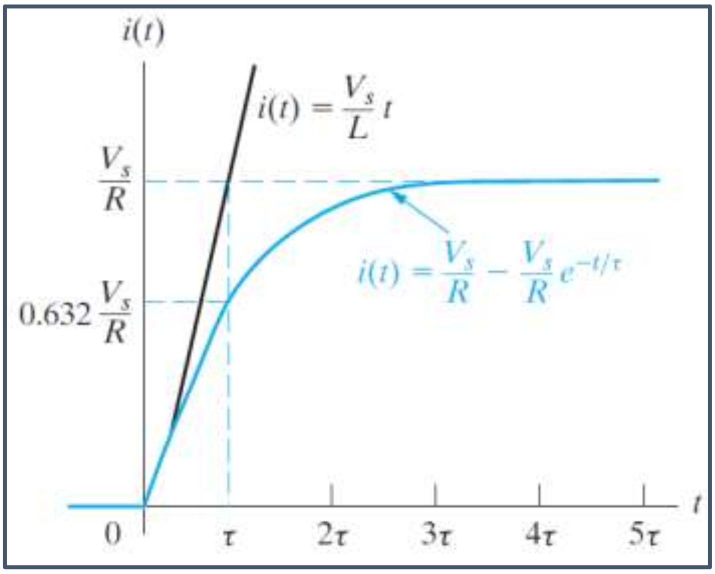

Figure 1.6: The step response of the RL circuit

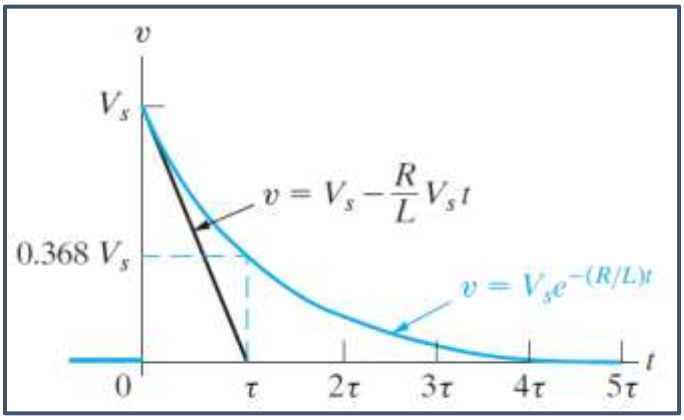

Figure 1.7: Inductor voltage versus time

Notes:

Complete response = transient response (temporary part) + steady-state response (permanent part)

- ***The transient response is the circuit's temporary response that will die out with time.***
- ***The steady-state response is the behavior of the circuit a long time after an external excitation is applied.***

$i(t) = i(\infty) + [i(0) - i(\infty)]e^{\frac{-t}{\tau}}$ Compare with the step response of the RL circuit

1.4 The Step Response of an RC Circuit

Consider the RC circuit in figure 1.8, summing the currents in the circuits:

$$C\frac{dv_C}{dt} + \frac{v_C}{R} = I_s.$$ (Dividing by C)

$$\frac{dv_C}{dt} + \frac{v_C}{RC} = \frac{I_s}{C}.$$

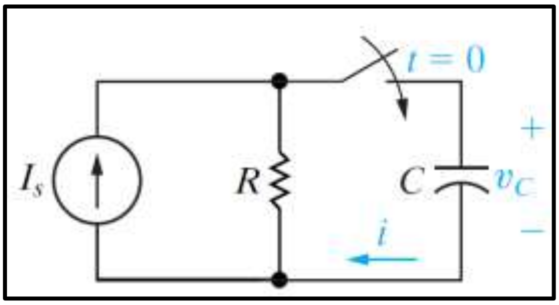

Figure 1.8

Comparing the above equation with the equation for the step response of the RL circuit reveals that the form of the solution for v_c is the same as that for the current in the inductive circuit.

The Step Response of an RC circuit is:

$$v_C = I_sR + (V_0 - I_sR)e^{-t/RC}, \quad t \geq 0.$$

A similar derivation for the current in the capacitor yields the differential equation:

$$\frac{di}{dt} + \frac{1}{RC}i = 0. \quad i = \left(I_s - \frac{V_0}{R}\right)e^{-t/RC}, \quad t \geq 0^+$$

V_0, is the initial value of the voltage across the capacitor

Example 1.1: The switch in the circuit shown in figure 1.9 has been closed for a long time before it is opened at t = 0. Find:

(a) $i_L(t)\ for\ t \geq 0$

(b) $i_o(t)\ for\ t \geq 0^+$

(c) $v_o(t)\ for\ t \geq 0$

(d) The percentage of the total energy stored in the 2 H inductor that is dissipated in the 10 Ω resistor

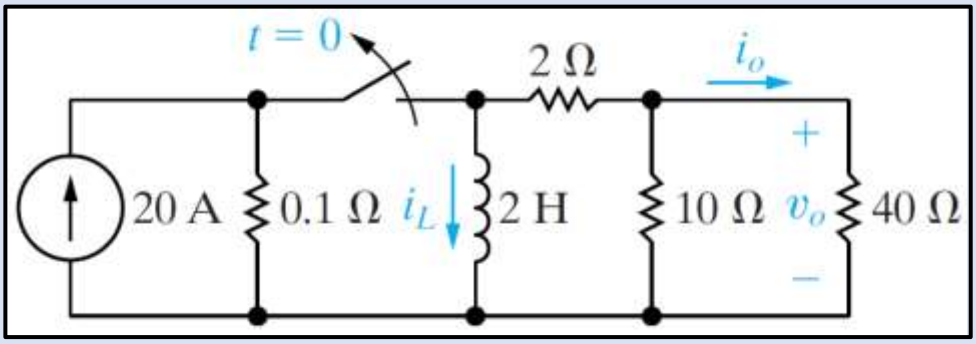

Figure 1.9

Solution:

a) $\text{Req} = 2 + 10||40 = 10\Omega$

The time constant of the circuit is L/ Req = 0.2 s

$$i_L(t) = 20e^{-5t}\ \text{A}, \quad t \geq 0.$$

b) using current division

$$i_o = -i_L \frac{10}{10 + 40}$$

The inductor behaves as a short circuit before the switch being opened, producing an instantaneous change in the current i_o. Then,

$$i_o(t) = -4e^{-5t}\ \text{A}, \quad t \geq 0^+$$

c) Using ohms law

$$v_o(t) = 40i_o = -160e^{-5t}\ \text{V}, \quad t \geq 0^+.$$

d) The power dissipated in the 10Ω resistor is:

$$p_{10\Omega}(t) = \frac{v_o^2}{10} = 2560e^{-10t}\ \text{W}, \quad t \geq 0^+.$$

The total energy dissipated in the 10Ω resistor is:

$$w_{10\Omega}(t) = \int_0^{\infty} 2560e^{-10t}\,dt = 256\text{ J}.$$

The initial energy stored in the 2 H inductor is

$$w(0) = \frac{1}{2}Li^2(0) = \frac{1}{2}(2)(400) = 400\text{ J}.$$

Therefore, the percentage of energy dissipated in the 10Ω resistor is:

$$\frac{256}{400}(100) = 64\%$$

Example 1.2: The switch in the circuit in figure 1.10 has been in position (a) for a long time. At $t = 0$ the switch moves from position (a) to position (b). The switch is a make-before-break type; that is, the connection at position b is established before the connection at position (a) is broken, so there is no interruption of current through the inductor.

a) Find the expression for $i(t)$ when $t \geq 0$

b) What is the initial voltage across the inductor just after the switch has been moved to position b?

c) How many milliseconds after the switch has been moved does the inductor voltage equal 24 V?

d) Plot both $i(t)$ and $v(t)$versus t.

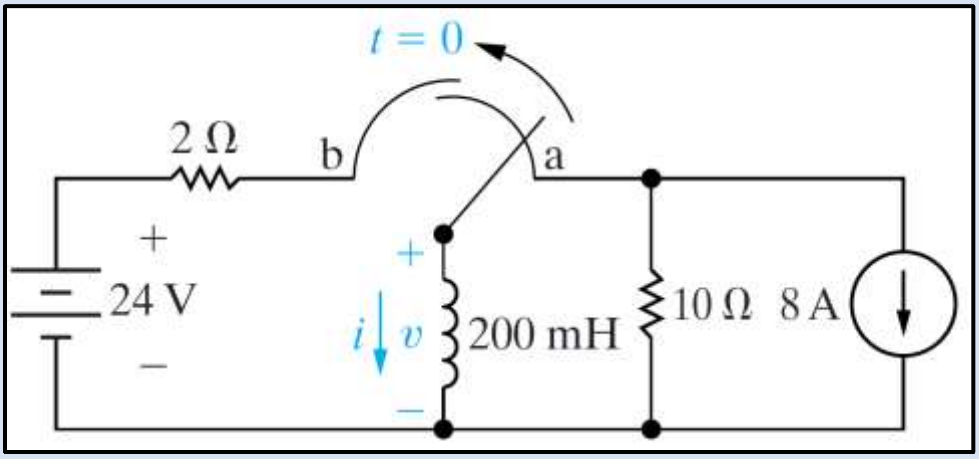

Figure 1.10

Solution:

a) When the switch is in position (a), the 200 mH inductor is a short circuit across the 8A current source. Therefore, the inductor carries an initial current of 8 A.

When the switch is in position (b), the final value of i will be 12 A

The time constant of the circuit is 100 ms.

$$i = 12 + (-8 - 12)e^{-t/0.1}$$
$$= 12 - 20e^{-10t}\text{ A}, \quad t \geq 0.$$

b) The voltage across the inductor is

$$v = L\frac{di}{dt}$$
$$= 0.2(200e^{-10t})$$
$$= 40e^{-10t}\text{ V}, \quad t \geq 0^{+}.$$

The initial inductor voltage is 40 V

c) We find the time at which the inductor voltage equals 24 V by solving the expression ($24 = 40e^{-10t}$)

$$t = \frac{1}{10}\ln\frac{40}{24}$$
$$= 51.08 \times 10^{-3}$$
$$= 51.08\text{ ms}.$$

d) Figure 1.11 shows the graphs of $i(t)$ and $v(t)$ versus t. Note that the instant of time when the current equals zero corresponds to the instant of time when the inductor voltage equals the source voltage of 24 V, as predicted by Kirchhoff's voltage law.

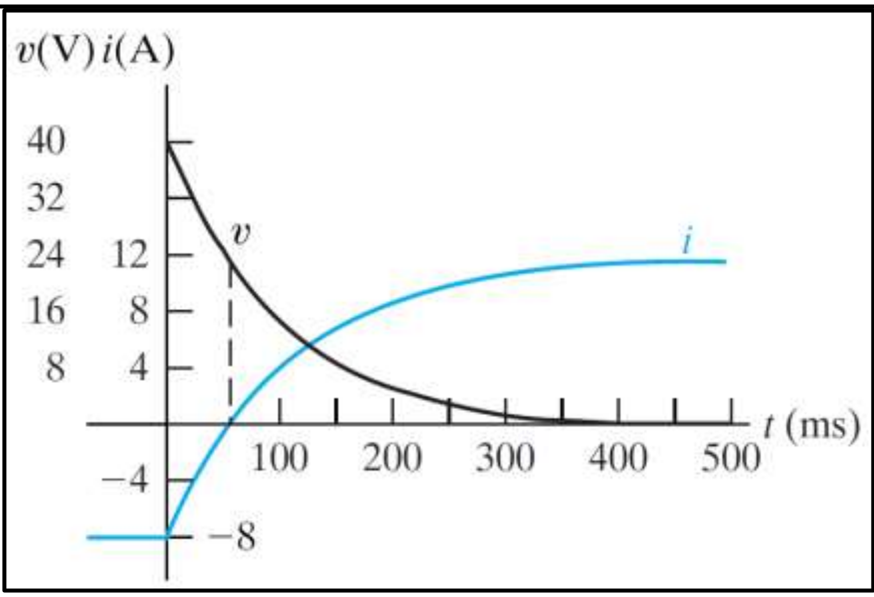

Figure 1.11

Example 1.3: The two switches in the circuit shown in figure 1.12 have been closed for a long time. At $t = 0$ switch 1 is opened. Then, 35 ms later, switch 2 is opened.

a) Find $i_L(t)\ for\ 0\ \leq t \leq 35ms$

b) Find $i_L(t)\ for\ t \geq 35ms$

c) What percentage of the initial energy stored in the 150 mH inductor is dissipated in the 18 Ω resistor?

d) Repeat (c) for the 3 Ω resistor.

e) Repeat (c) for the 6 Ω resistor

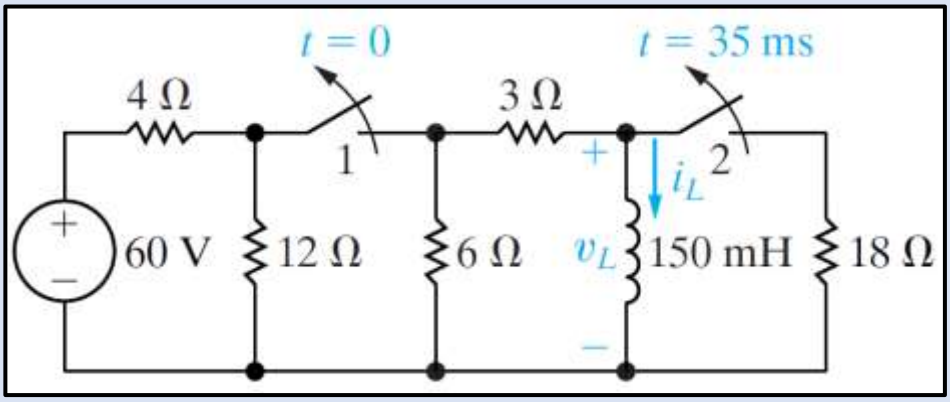

Figure 1.12

Solution:

a) For t < 0 both switches are closed, causing the 150 mH inductor to short-circuit the resistor as shown in figure 1.13.

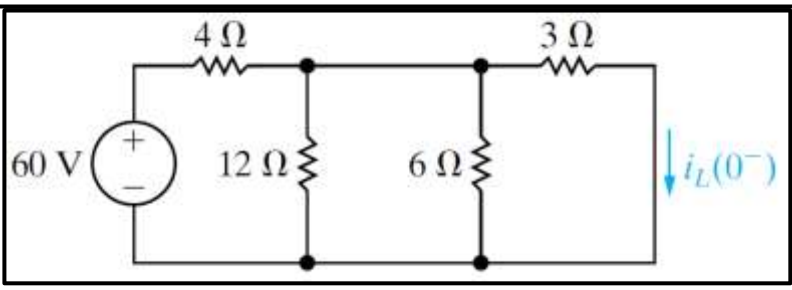

Figure 1.13

For $0 \leq t \leq 35ms$ switch 1 is open (switch 2 is closed), which disconnects the 60 V voltage source and the 4 Ω and 12 Ω resistors from the circuit, the resultant circuit is shown in figure 1.14.

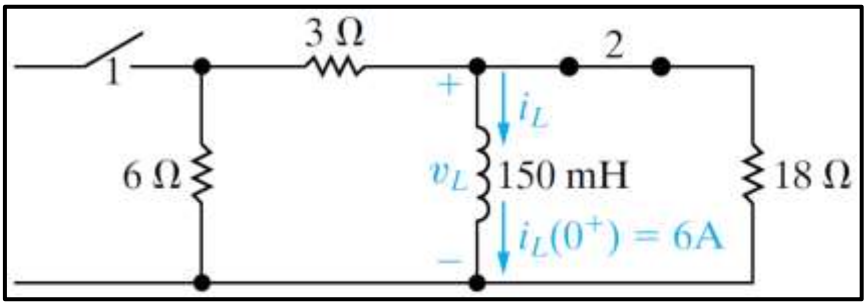

Figure 1.14

$Req = 9||18 = 6\,\Omega$, the time constant of the circuit is $\left(\frac{150}{6}\right) * 10^{-3} = 25\ ms$

$i_L = 6e^{-40t}$ A, $\quad 0 \leq t \leq 35$ ms.

b) When t > 35ms the value of the inductor current is:

$i_L = 6e^{-1.4} = 1.48\ A$

Thus, when switch 2 is opened, the circuit reduces to the one shown in figure 1.15.

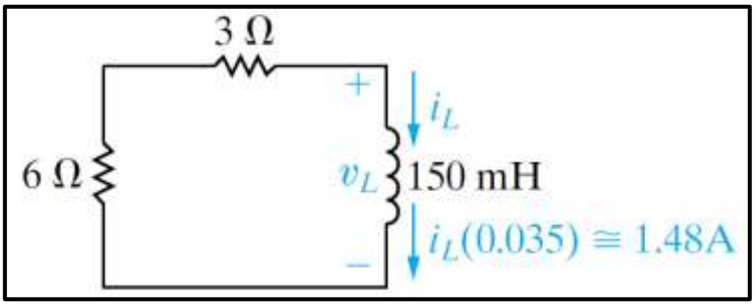

Figure 1.15

The time constant changes to $\left(\frac{150}{9}\right) * 10^{-3} = 16.67\ ms$

$$i_L = 1.48e^{-60(t-0.035)}\ \text{A}, \quad t \geq 35\ \text{ms}.$$

c) The 18 Ω resistor is in the circuit only during the first 35 ms of the switching sequence. During this interval, the voltage across the resistor is:

$$v_L = 0.15\frac{d}{dt}(6e^{-40t})$$

$$= -36e^{-40t}\ \text{V}, \quad 0 < t < 35\ \text{ms}.$$

The power dissipated in the 18 Ω resistor is:

$$p = \frac{v_L^2}{18} = 72e^{-80t}\ \text{W}, \quad 0 < t < 35\ \text{ms}.$$

Hence the energy dissipated is:

$$w = \int_0^{0.035} 72e^{-80t}\,dt \quad = \frac{72}{-80}e^{-80t}\Big|_0^{0.035}$$

$$= 0.9(1 - e^{-2.8})$$

$$= 845.27\ \text{mJ}.$$

The initial energy stored in the 150 mH inductor is:

$$w_i = \frac{1}{2}(0.15)(36) = 2.7\ \text{J} = 2700\ \text{mJ}.$$

Therefore, 31.31% of the initial energy stored in the 150 mH inductor is dissipated in the 18 Ω resistor.

d) For $0 \leq t \leq 35ms$ the voltage across the 3 Ω resistor is:

$$v_{3\Omega} = \left(\frac{v_L}{9}\right)(3) = \frac{1}{3}v_L = -12e^{-40t}\ V$$

Therefore, the energy dissipated in the 3 Ω resistor in the first 35 ms is:

$$w_{3\Omega} = \int_0^{0.035} \frac{144e^{-80t}}{3}dt$$

$$= 0.6(1 - e^{-2.8})$$

$$= 563.51\ \text{mJ}.$$

For $t \geq 35ms$ the current in the 3 Ω resistor is:

$$i_{3\Omega} = i_L = (6e^{-1.4})e^{-60(t-0.035)} \text{ A.}$$

Hence the energy dissipated in the 3 Ω resistor for $t \geq 35ms$ is:

$$w_{3\Omega} = \int_{0.035}^{\infty} i_{3\Omega}^2 \times 3\,dt$$

$$= \int_{0.035}^{\infty} 3(36)e^{-2.8}e^{-120(t-0.035)}\,dt$$

$$= 108e^{-2.8} \times \left.\frac{e^{-120(t-0.035)}}{-120}\right|_{0.035}^{\infty}$$

$$= \frac{108}{120}e^{-2.8} = 54.73 \text{ mJ.}$$

The total energy dissipated in the 3 Ω resistor is:

$$w_{3\Omega}(\text{total}) = 563.51 + 54.73$$

$$= 618.24 \text{ mJ.}$$

The percentage of the initial energy stored is:

$$\frac{618.24}{2700} * 100 = 22.90\%$$

e) Because the 6 Ω resistor is in series with the 3 Ω resistor, the energy dissipated and the percentage of the initial energy stored will be twice that of the 3 Ω resistor:

$w_{6\Omega}(total) = 1236.48$ mJ

And the percentage of the initial energy stored is 45.80%.

1236.48 + 618.24 + 845.27 = 2699.99 mJ

31.31 + 22.90 + 45.80 = 100.01%.

1.5 Natural Response of Series RLC Circuits

Consider the series RLC circuit shown in figure 1.16. The circuit is being excited by the energy initially stored in the capacitor and inductor.

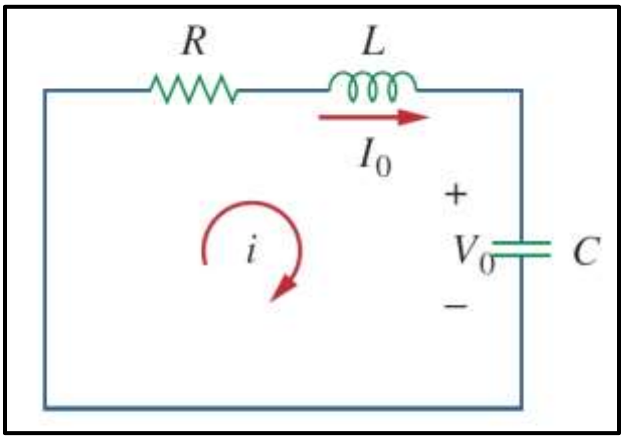

Figure 1.16

The energy is represented by the initial capacitor voltage V_0 and initial inductor current I_0. Thus, at t = 0,

$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0$$

$$i(0) = I_0$$

Applying KVL around the loop

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)d\tau = 0$$

To eliminate the integral, we differentiate with respect to t and rearrange terms. We get:

$$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

(Second-order differential equation)

To solve the above equation, we need two initial conditions such as the initial value of i and its first derivative or initial values of some i and v. The initial value of the derivative of i:

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0 \qquad \frac{di(0)}{dt} = -\frac{1}{L}(RI_0 + V_0)$$

Let $i = Ae^{st}$, A and s are constants to be determined

$$As^2e^{st} + \frac{AR}{L}se^{st} + \frac{A}{LC}e^{st} = 0$$

$$Ae^{st}\left(s^2 + \frac{R}{L}s + \frac{1}{LC}\right) = 0$$

$$s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

This quadratic equation is known as the characteristic equation of the Second-order differential. The two roots of the above equation are:

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}, \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{R}{2L}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- The roots s_1 and s_2 are called natural frequencies measured in nepers per second (Np/s) because they are associated with the natural response of the circuit.
- ω_0 is known as the resonant frequency or strictly as the undamped natural frequency, expressed in radians per second (rad/s).
- α is the neper frequency or the damping factor, expressed in nepers per second.

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

There are two possible solutions for *i*

$$i_1 = A_1e^{s_1t}, \qquad i_2 = A_2e^{s_2t}$$

Thus, the natural response of the series RLC circuit is:

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

The constants A_1 and A_2 are determined from the initial values $i(0)$ and $\frac{di(0)}{dt}$

a) If $\alpha > \omega_0$, we have the <u>overdamped</u> case.

b) If $\alpha = \omega_0$, we have the <u>critically damped</u> case.

c) If $\alpha < \omega_0$, we have the <u>underdamped</u> case

<u>Overdamped Case ($\alpha > \omega_0$)</u>

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

<u>Critically Damped Case ($\alpha = \omega_0$)</u>

$$i(t) = (A_2 + A_1 t)e^{-\alpha t}$$

<u>Underdamped Case ($\alpha < \omega_0$)</u>

$$s_1 = -\alpha + \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha + j\omega_d$$

$$s_2 = -\alpha - \sqrt{-(\omega_0^2 - \alpha^2)} = -\alpha - j\omega_d$$

- $j = \sqrt{-1}$
- $\omega_d = \sqrt{{\omega_0}^2 - \alpha^2}$: is called the damping natural frequency.
- ω_0: the undamped natural frequency

The natural response is:

$$\begin{aligned} i(t) &= A_1 e^{-(\alpha - j\omega_d)t} + A_2 e^{-(\alpha + j\omega_d)t} \\ &= e^{-\alpha t}(A_1 e^{j\omega_d t} + A_2 e^{-j\omega_d t}) \end{aligned}$$

Using Euler's identities

$$e^{j\theta} = \cos\theta + j\sin\theta, \qquad e^{-j\theta} = \cos\theta - j\sin\theta$$

$$\begin{aligned} i(t) &= e^{-\alpha t}[A_1(\cos\omega_d t + j\sin\omega_d t) + A_2(\cos\omega_d t - j\sin\omega_d t)] \\ &= e^{-\alpha t}[(A_1 + A_2)\cos\omega_d t + j(A_1 - A_2)\sin\omega_d t] \end{aligned}$$

Replacing $(A_1 + A_2)$ constants and $j\,(A_1 - A_2)$ with constants B_1 and B_2

$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

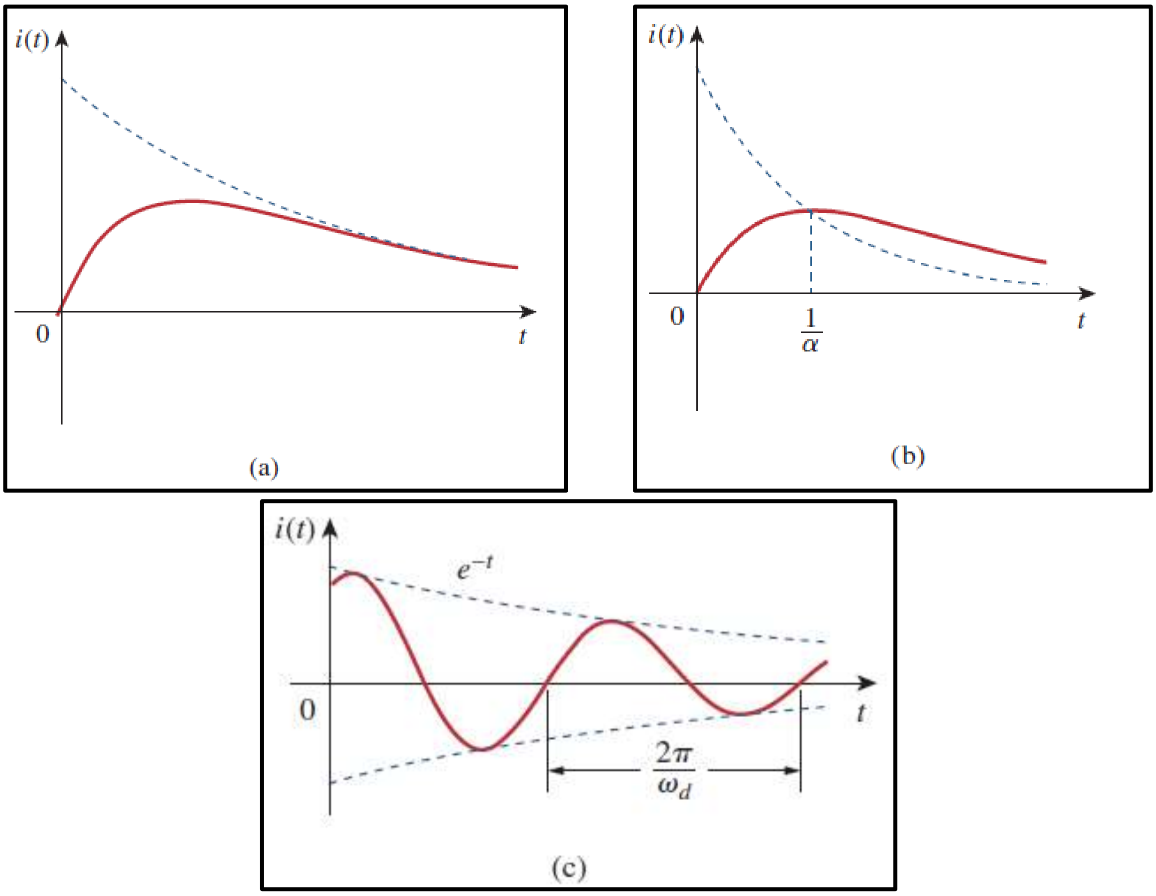

Figure 1.17; (a) Overdamped response, (b) critically damped response, (c) underdamped response.

Example 1.4: Find $i(t)$ in the circuit of Fig. 1.18. Assume that the circuit has reached a steady-state at $t = 0$.

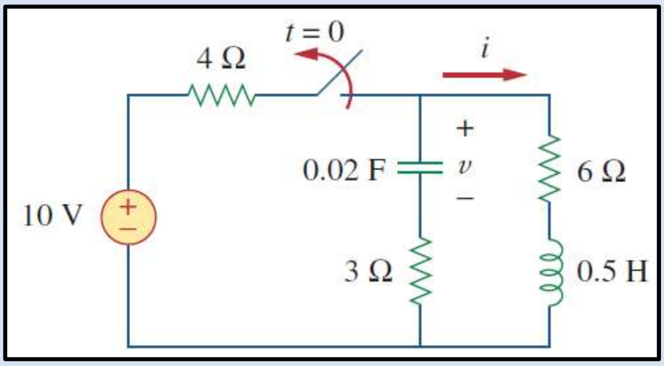

Figure 1.18

Solution:

For t < 0, the switch is closed. The capacitor acts like an open circuit while the inductor acts like a shunted circuit. The equivalent circuit is shown in figure

1.19 (a). Thus, at t = 0,

$$i(0) = \frac{10}{4+6} = 1 \text{ A}, \qquad v(0) = 6i(0) = 6 \text{ V}$$

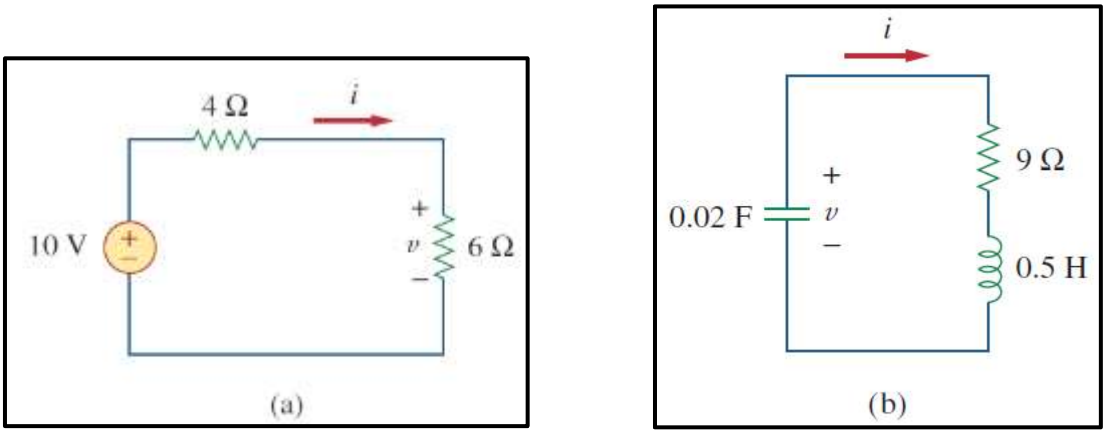

Figure 1.19: (a) for t < 0, (b) for t > 0.

Where $i(0)$ is the initial current through the inductor and is the initial voltage across the capacitor. For t > 0 the switch is opened and the voltage source is disconnected. The equivalent circuit is shown in figure 1.19 (b) which is a source free series *RLC* circuit. Notice that the 3 Ω and 6 Ω resistors, which are in series in figure 1.19 when the switch is opened, have been combined to give R = 9 Ω in figure 1.19 (b). The roots are calculated as follows:

$$\alpha = \frac{R}{2L} = \frac{9}{2(\frac{1}{2})} = 9, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\frac{1}{2} \times \frac{1}{50}}} = 10$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -9 \pm \sqrt{81 - 100}$$

$$s_{1,2} = -9 \pm j4.359$$

Hence, the response is underdamped ($\alpha < \omega$); that is,

$$\boxed{i(t) = e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)} \qquad \text{eq. (a)}$$

Now obtain A1 and A2 using the initial conditions. At t = 0;

$$i(0) = 1 = A_1$$

$$\boxed{\left.\frac{di}{dt}\right|_{t=0} = -\frac{1}{L}[Ri(0) + v(0)] = -2[9(1) - 6] = -6 \text{ A/s}} \qquad \text{eq. (b)}$$

Note that $v(0) = V_0 = -6V$ is used, because the polarity of *v* in figure 1.19

(b) is opposite that in figure 1.16. Taking the derivative of $i(t)$ in equation (a):

$$\frac{di}{dt} = -9e^{-9t}(A_1 \cos 4.359t + A_2 \sin 4.359t)$$
$$+ e^{-9t}(4.359)(-A_1 \sin 4.359t + A_2 \cos 4.359t)$$

Imposing the condition in equation (b) at t = 0 gives:

$$-6 = -9(A_1 + 0) + 4.359(-0 + A_2)$$

But A1 = 1. Then

$$-6 = -9 + 4.359A_2 \qquad \Rightarrow \qquad A_2 = 0.6882$$

Substituting the values of A1 and A2 in equation (a) yields the complete solution as:

$$i(t) = e^{-9t}(\cos 4.359t + 0.6882 \sin 4.359t) \text{ A}$$

1.6 Natural Response of Parallel RLC Circuits

Consider the parallel RLC circuit shown in figure 1.20. Assume initial inductor current I_0 and initial capacitor voltage V_0

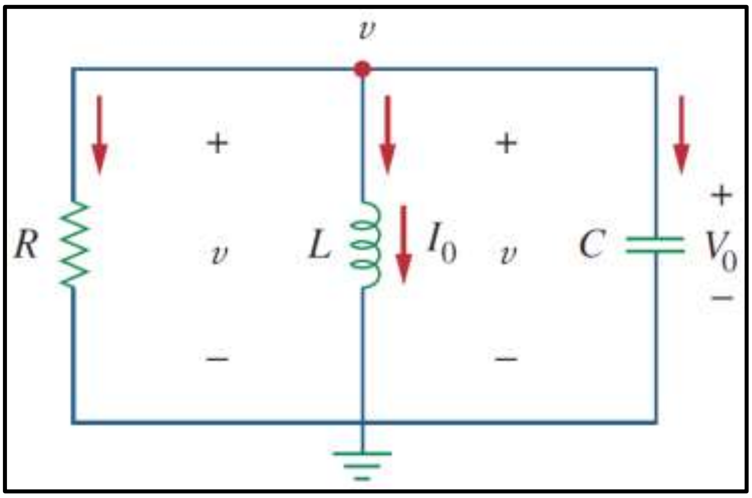

Figure 1.20

$$i(0) = I_0 = \frac{1}{L}\int_{\infty}^{0} v(t)\,dt$$

$$v(0) = V_0$$

Applying KCL at the top node gives

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau + C\frac{dv}{dt} = 0$$

Taking the derivative with respect to t and dividing it by C results in

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

The roots of the characteristic equation are

$$s_{1,2} = -\frac{1}{2RC} \pm \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC}, \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

Overdamped Case ($\alpha > \omega_0$)

$$v(t) = A_1e^{s_1t} + A_2e^{s_2t}$$

Critically Damped Case ($\alpha = \omega_0$)

$$v(t) = (A_1 + A_2t)e^{-\alpha t}$$

Underdamped Case ($\alpha < \omega_0$)

$$v(t) = e^{-\alpha t}(A_1 \cos \omega_d t + A_2 \sin \omega_d t)$$

The constants A_1 and A_2 in each case can be determined from the initial conditions. We need $v(0)$ and $\frac{dv(0)}{dt}$.

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0 \qquad \frac{dv(0)}{dt} = -\frac{(V_0 + RI_0)}{RC}$$

Example 1.5: In the parallel circuit of figure 1.21, find $v(t)$ for $t > 0$, assuming $v(0) = 5V, i(0) = 0A, L = 1H,$ and $C = 10mF$. Consider these cases: $R = 1.923\ \Omega, R = 5\ \Omega,$ and $R = 6.25\ \Omega$

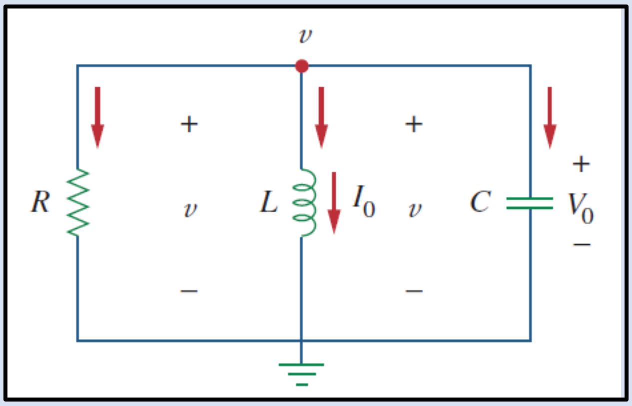

Figure 1.21

Solution:

Case 1: If $R = 1.923\ \Omega$

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 1.923 \times 10 \times 10^{-3}} = 26$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 10 \times 10^{-3}}} = 10$$

Since $\alpha > \omega_0$ in this case, the response is *overdamped*. The roots of the characteristic's equation are:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2, -50$$

And the corresponding response is:

$$v(t) = A_1 e^{-2t} + A_2 e^{-50t}$$

Applying the initial conditions to get A1 and A2:

$$v(0) = 5 = A_1 + A_2$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{1.923 \times 10 \times 10^{-3}} = -260$$

Differentiating $v(t)$

$$\frac{dv}{dt} = -2A_1 e^{-2t} - 50A_2 e^{-50t}$$

$$-260 = -2A_1 - 50A_2$$

Then A1 $= -0.2083$ and A2 $= 5.208$

Substituting A1 and A2 in the v(t) equation yields:

$v(t) = -0.2083\, e^{-2t} + 5.208\, e^{-50t}$ V

Case 2: R = 5 Ω

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 10 \times 10^{-3}} = 10$$

While ω_0 remains the same. Since $\alpha = \omega_0 = 10$, the response is *critically damped*. Hence, $S1 = S2 = -10$ and

$$v(t) = (A_1 + A_2 t)e^{-10t}$$

Applying the initial conditions to get A1 and A2:

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{5 \times 10 \times 10^{-3}} = -100$$

Differentiating v(t):

$$\frac{dv}{dt} = (-10A_1 - 10A_2 t + A_2)e^{-10t}$$

At $t = 0$,

$-100 = -10A1 + A2$

$A1 = 5$ and $A2 = -50$, thus:

$v(t) = (5 - 50t)e^{-10t}$ V

Case 3: R = 6.25 Ω

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 6.25 \times 10 \times 10^{-3}} = 8$$

While ω_0 remains the same. As $\alpha < \omega_0$ in this case, the response is *underdamped*. The roots of the characteristic equation are

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -8 \pm j6$$

Then,

$$v(t) = (A_1 \cos 6t + A_2 \sin 6t)e^{-8t}$$

Now obtain A1 and A2 as

$$v(0) = 5 = A_1$$

$$\frac{dv(0)}{dt} = -\frac{v(0) + Ri(0)}{RC} = -\frac{5 + 0}{6.25 \times 10 \times 10^{-3}} = -80$$

Differentiating $v(t)$

$$\frac{dv}{dt} = (-8A_1 \cos 6t - 8A_2 \sin 6t - 6A_1 \sin 6t + 6A_2 \cos 6t)e^{-8t}$$

At $t = 0$,

$-80 = -8\ A1 + 6\ A2;$

$A1 = 5$ and $A2 = -6.667$

$$\mathbf{v(t) = (5\ cos6t - 6.667\ sin6t)e^{-8t}\ \ V}$$

Note; by increasing the value of R, the degree of damping decreases, and the response differ as shown in figure 1.22.

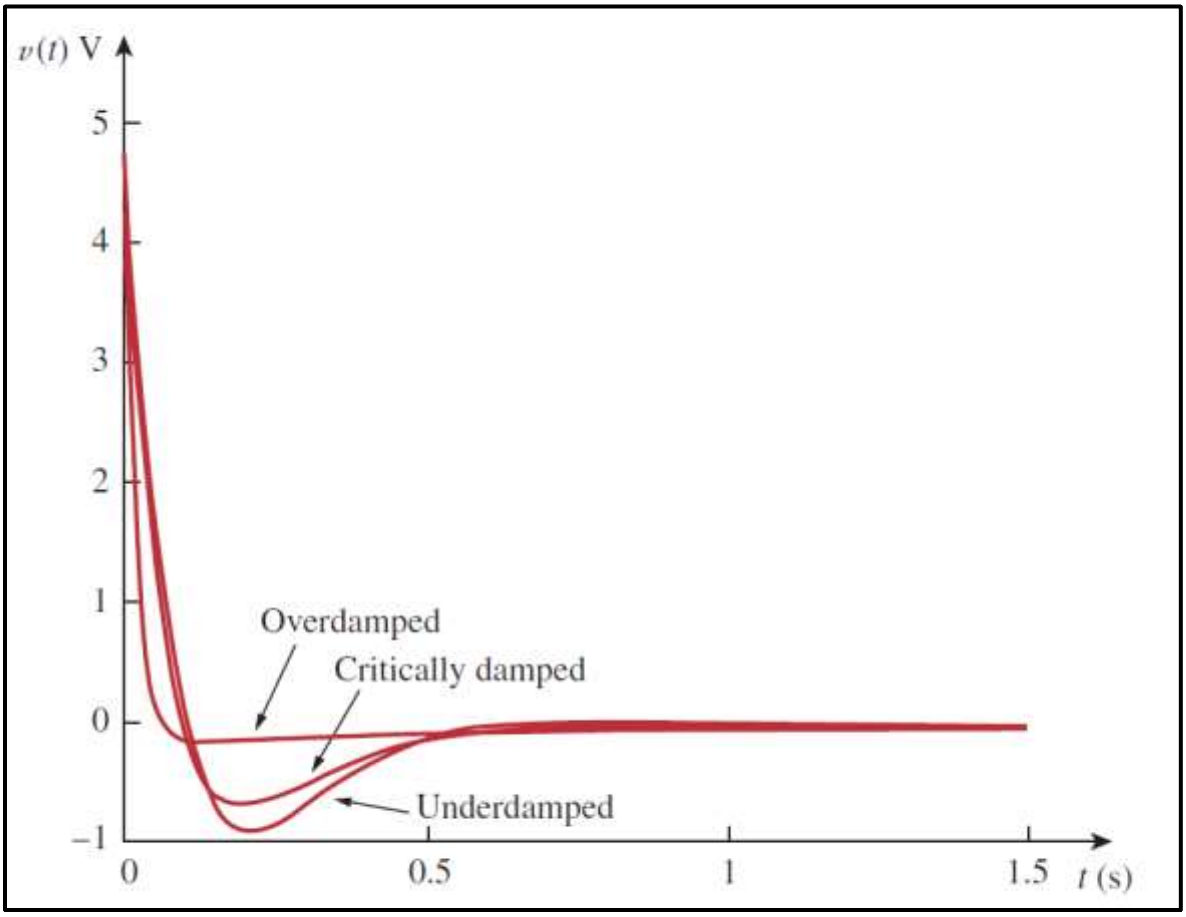

Figure 1.22

1.7 Step Response of Series RLC Circuits

Consider the series RLC circuit shown in figure 1.23

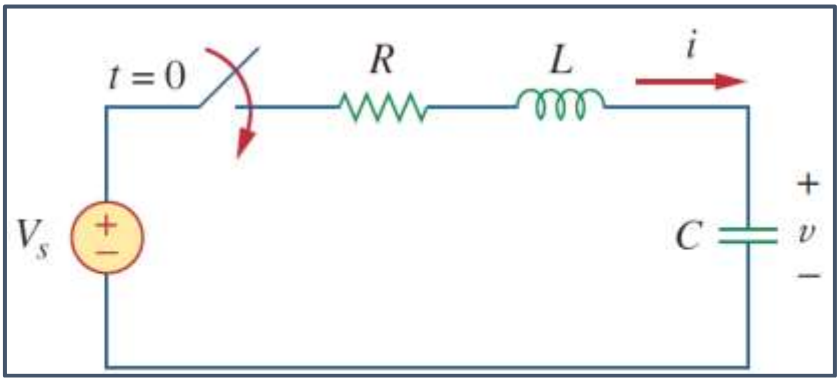

Figure 1.23

Applying KVL around the loop for t > 0,

$$L\frac{di}{dt} + Ri + v = V_s, \qquad i = C\frac{dv}{dt}$$

$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{v}{LC} = \frac{V_s}{LC}$$

Which has the same form as the equation of the natural response of the series RLC. The solution to the above equation has two components: the transient response and the steady-state response that is:

$$v(t) = v_t(t) + v_{ss}(t)$$

The transient response is the component of the total response that dies out with time. The form of the transient response is the same as the form of the solution obtained in Section 1.5 for the natural response circuit. Therefore, the transient response for the overdamped, underdamped, and critically damped cases are:

$$v_t(t) = A_1e^{s_1t} + A_2e^{s_2t} \qquad \text{(Overdamped)}$$

$$v_t(t) = (A_1 + A_2t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$v_t(t) = (A_1\cos\omega_d t + A_2\sin\omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

The steady-state response is the final value of $v(t)$. In the circuit in figure 1.23, the final value of the capacitor voltage is the same as the source voltage V_s. Hence,

$$v_{ss} = v(\infty) = V_s$$

Thus, the complete solutions for the overdamped, underdamped, and critically damped cases are:

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)}$$

$$v(t) = V_s + (A_1 + A_2 t)e^{-\alpha t} \quad \text{(Critically damped)}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \quad \text{(Underdamped)}$$

Example 1.6: For the circuit in figure 1.24, find $v(t)$ and $i(t)$ for $t > 0$. Consider these cases: $R = 5\ \Omega$, $R = 4\ \Omega$ *and* $R = 1\ \Omega$

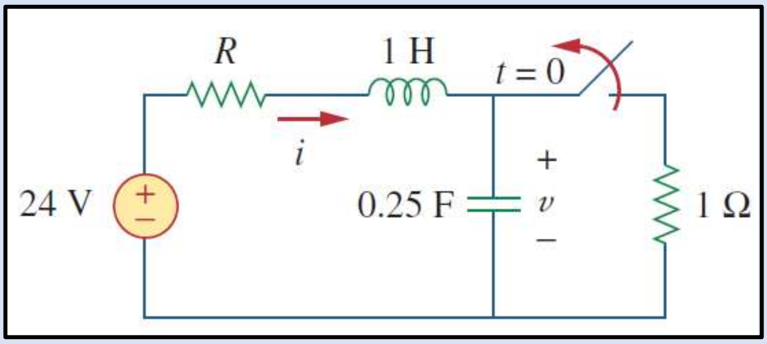

Figure 1.24

Solution:

Case 1: $R = 5\ \Omega$

For $t < 0$ the switch is closed for a long time. The capacitor behaves like an open circuit while the inductor acts like a short circuit. The initial current through the inductor is:

$$i(0) = \frac{24}{5+1} = 4 \text{ A}$$

The initial voltage across the capacitor is the same as the voltage across the 1 Ω resistor; that is, $v(0) = 1i(0) = 4V$

For $t > 0$ the switch is opened so that we have the 1 Ω resistor disconnected. What remains is the series *RLC* circuit with the voltage source. The characteristic roots are determined as follows:

$$\alpha = \frac{R}{2L} = \frac{5}{2 \times 1} = 2.5, \qquad \omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{1 \times 0.25}} = 2$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -1, -4$$

Since $\alpha > \omega_0$ we have the overdamped natural response. The total response is therefore

$$v(t) = v_{ss} + (A_1e^{-t} + A_2e^{-4t})$$

Where, v_{ss} is the steady-state response. It is the final value of the capacitor voltage. In figure 1.24, $v_{ss} = 24\ V$ Thus

$$v(t) = 24 + (A_1e^{-t} + A_2e^{-4t})$$

To find A1 and A2 using the initial conditions:

$v(0) = 4 = 24 + A1 + A2$ $\quad -24 = A1 + A2$

The current through the inductor cannot change abruptly and is the same current through the capacitor at $t = 0$ because the inductor and capacitor are now in series. Hence:

$$\frac{dv}{dt} = -A_1e^{-t} - 4A_2e^{-4t}$$

$$\frac{dv(0)}{dt} = 16 = -A_1 - 4A_2$$

$A1 = -64/3\ and\ A2 = 4/3$. Substituting in the voltage response equation:

$$v(t) = 24 + \frac{4}{3}(-16e^{-t} + e^{-4t})\ \text{V}$$

Since the inductor and capacitor are in series for $t > 0$, the inductor current is the same as the capacitor current. Then:

$$i(t) = C\frac{dv}{dt}$$

$$i(t) = \frac{4}{3}(4e^{-t} - e^{-4t})\ A$$

Case 2:($R = 4\ \Omega$) & Case 3: ($R = 1\ \Omega$) H.W

Ans:

$$i(t) = (4.8 + 9.6t)e^{-2t}\ \text{A}$$

$$i(t) = (3.1\ \sin 1.936t + 12\ \cos 1.936t)e^{-0.5t}\ \text{A}$$

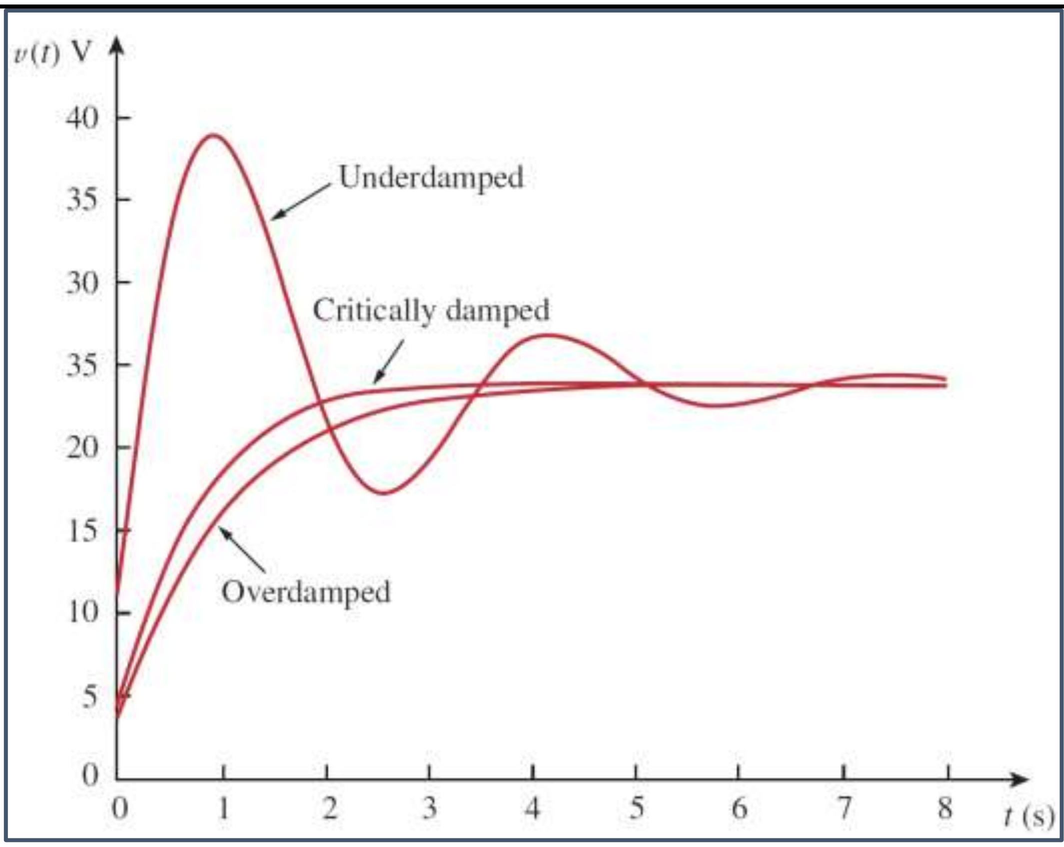

Figure 1.25

1.8 Step Response of Parallel RLC Circuits

Consider the parallel RLC circuit shown in figure 1.26:

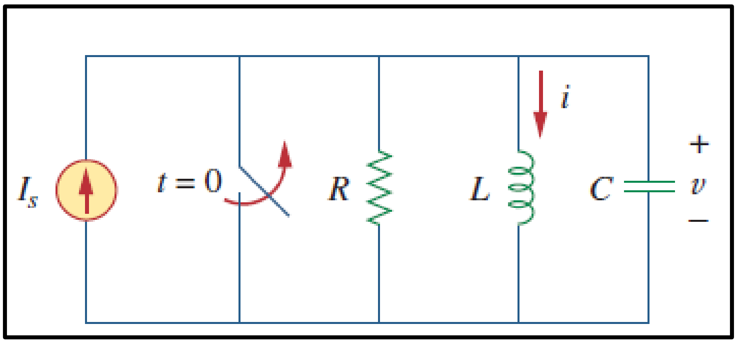

Figure 1.26

Applying KCL at the top node for t > 0

$$\frac{v}{R} + i + C\frac{dv}{dt} = I_s \quad v = L\frac{di}{dt}$$

$$\frac{d^2i}{dt^2} + \frac{1}{RC}\frac{di}{dt} + \frac{i}{LC} = \frac{I_s}{LC}$$

The solution to the above equation has two components: the transient response and the steady-state response that is:

$$i(t) = i_t(t) + i_{ss}(t)$$

$$i(t) = I_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad \text{(Overdamped)}$$

$$i(t) = I_s + (A_1 + A_2 t)e^{-\alpha t} \qquad \text{(Critically damped)}$$

$$i(t) = I_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t)e^{-\alpha t} \qquad \text{(Underdamped)}$$

The constants A1 and A2 in each case can be determined from the initial conditions for i *and* di/dt. The equation above only applies for finding the inductor current $i_L = i$. But once the inductor current is known, we can find $v = Ldi/dt$ that is the same voltage across the inductor, capacitor, and resistor.

Example 1.7: In the circuit of figure 1.27, find $i(t)$ *and* $i_R(t)$ *for* $t > 0$.

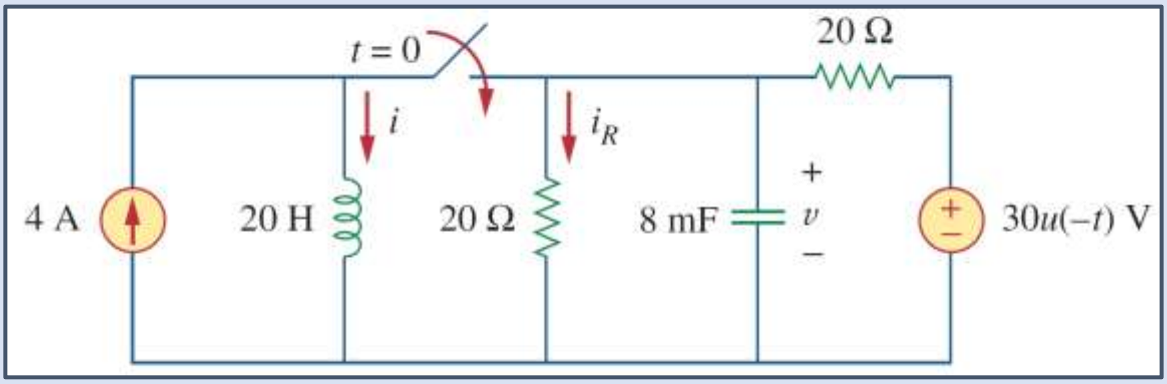

Figure 1.27

Here, $u(t)$: unit step function

$$u(t) = \begin{cases} 1, t > 0 \\ 0, t < 0 \end{cases}$$

Solution:

For $t < 0$ the switch is open, and the circuit is partitioned into two independent sub-circuits. The 4-A current flows through the inductor, so that $i(0) = 4\ A$. Since $30u(-t) = 30$ when $t < 0$ and 0 when $t > 0$ the voltage source is operative for $t < 0$. The capacitor acts like an open circuit and the voltage across it is the same as the voltage across the 20 Ω resistor connected in parallel with it. By voltage division, the initial capacitor voltage is:

$$v(0) = \frac{20}{20 + 20}(30) = 15\ \text{V}$$

For $t > 0$ the switch is closed, and we have a parallel *RLC* circuit with a current source. The voltage source is zero which means it acts like a short circuit. The two 20 Ω resistors are now in parallel. They are combined to give $R = 10\ \Omega$. The characteristic roots are determined as follows:

$$\alpha = \frac{1}{2RC} = \frac{1}{2 \times 10 \times 8 \times 10^{-3}} = 6.25$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{20 \times 8 \times 10^{-3}}} = 2.5$$

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -6.25 \pm \sqrt{39.0625 - 6.25}$$
$$= -6.25 \pm 5.7282$$

$s_1 = -11.978\ and\ s_2 = -0.5218$

Since $\alpha > \omega_0$ we have the overdamped case. Hence:

$$i(t) = I_s + A_1 e^{-11.978t} + A_2 e^{-0.5218t}$$

$Is = 4\ A$, is the final value of $i(t)$. Using the initial conditions to determine A1 and A2. At $t = 0$:

$i(0) = 4 = 4 + A1 + A2,$ then, $A1 = -A2$

Taking the derivative of $i(t)$

$$\frac{di}{dt} = -11.978 A_1 e^{-11.978t} - 0.5218 A_2 e^{-0.5218t}$$

So that at $t = 0$;

$$\frac{di(0)}{dt} = -11.978 A_1 - 0.5218 A_2$$

$$L\frac{di(0)}{dt} = v(0) = 15 \quad \Rightarrow \quad \frac{di(0)}{dt} = \frac{15}{L} = \frac{15}{20} = 0.75$$

$$0.75 = (11.978 - 0.5218)A_2 \quad \Rightarrow \quad A_2 = 0.0655$$

Thus $A1 = -0.0655\ and\ A2 = 0.0655$. Substituting A1 and A2 gives the complete solution:

$$i(t) = 4 + 0.0655(e^{-0.5218t} - e^{-11.978t})\ \text{A}$$

From $i(t)$, we can obtain $v(t) = Ldi/dt$

$$i_R(t) = \frac{v(t)}{20} = \frac{L}{20}\frac{di}{dt} = 0.785 e^{-11.978t} - 0.0342 e^{-0.5218t}\ \text{A}$$

Homework

H.W 1.1: In the circuit in figure 1.28, the switch has been closed for a long time before opening at $t = 0$.

a) Find the value of L so that $v_o(t)$ equals 0.5 $v_o(0)$ when $t = 1$ ms

b) Find the percentage of the stored energy that has been dissipated in the 10-Ω resistor when $t = 1$ ms

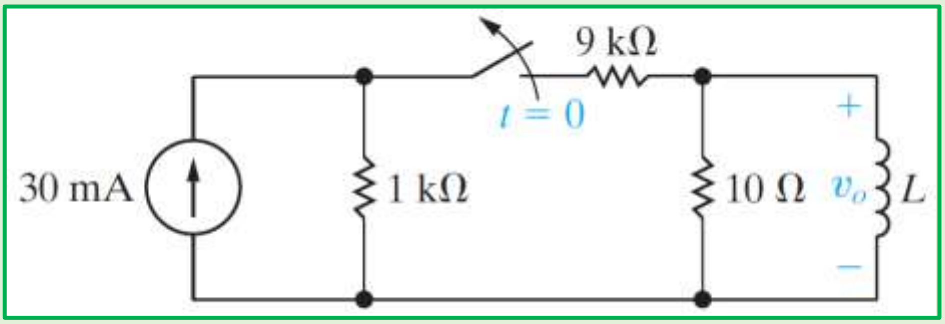

Figure 1.28

Ans: (a) L= 14.43mH, (b) 75 %

H.W 1.2: The switch in the circuit in figure 1.29 has been in the left position for a long time. At $t = 0$ it moves to the right position and stays there.

a) Write the expression for the capacitor voltage, $v(t)$ for $t \geq 0$

b) Write the expression for the current through the 40-k Ω resistor, $i(t)$ for $t \geq 0$,

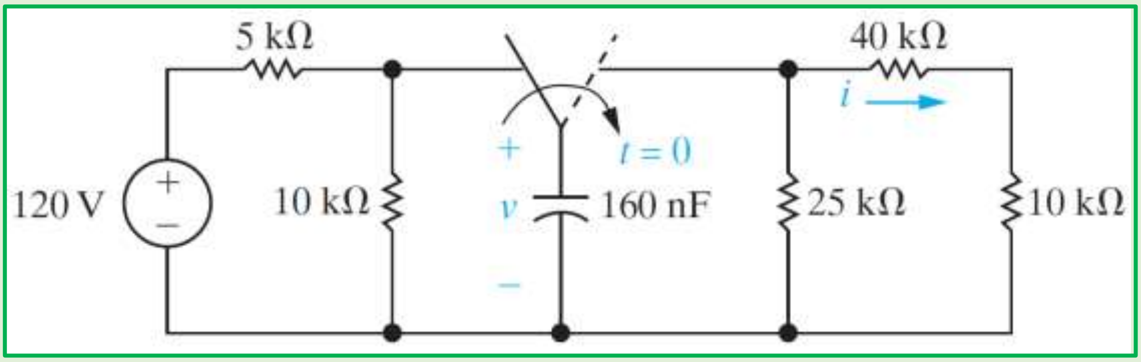

Figure 1.29

Ans: $\boldsymbol{v(t) = (80e^{-375t})\,V,\ t \geq 0,}$ $\boldsymbol{i(t) = (1.6e^{-375t})\,mA,\ t \geq 0}$

H.W 1.3: The current and voltage at the terminals of the inductor in the circuit in figure 1.30 are:

$i(t) = (4 + 4e^{-40t})$ A, $\quad t \geq 0$

$v(t) = (-80e^{-40t})$ V, $\quad t \geq 0^+$

a) Specify the numerical values of *Vs*, *R*, *Io*, and *L*.

b) How many milliseconds after the switch has been closed does the energy stored in the inductor reach 9 J?

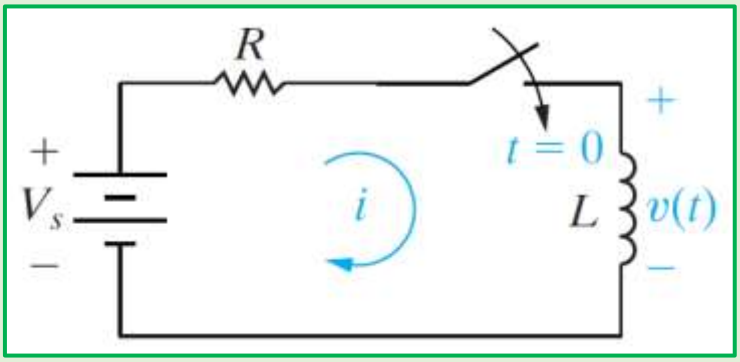

Figure 1.30

Ans: Vs=80 V, R=20-Ω, Io=8A, L=0.5H and t = 17.33ms

H.W 1.4: The switch in the circuit seen in figure 1.31 has been in position (a) for a long time. At $t = 0$ the switch moves instantaneously to position (b). For $t \geq 0$, find

a) $v_o(t)$

b) $i_o(t)$

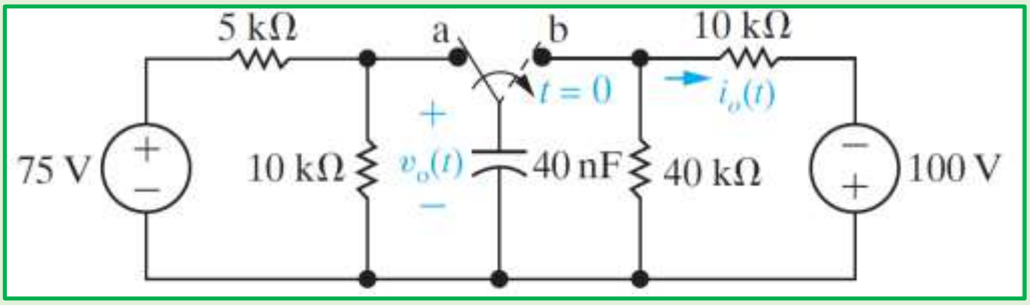

Figure 1.31

Ans: $\boldsymbol{v_o(t) = -80 + 130e^{-3125t}\ V,\ i_o(t) = 13e^{-3125t} + 2\ mA}$

H.W 1.5: The switch in the circuit shown in figure 1.32 has been closed for a long time. The switch opens at $t = 0$. Find $v_o(t)$ for $t \geq 0$

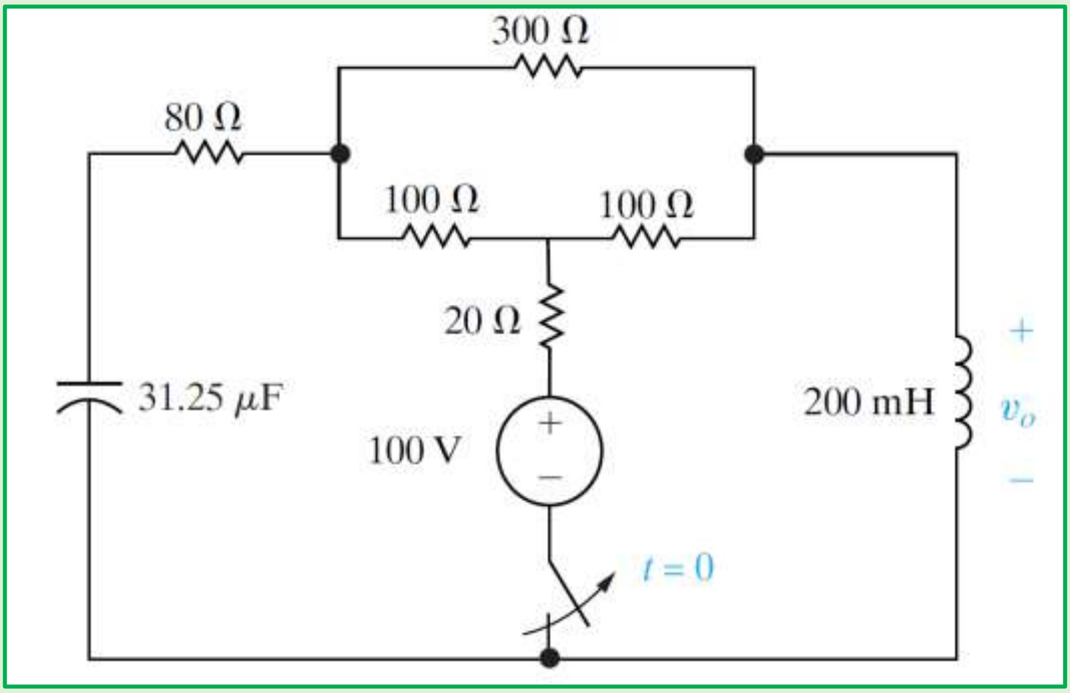

Figure 1.32

Ans: $v_o(t) = -6.67e^{-200t} - 133.33\, e^{-800t}\ V, \qquad t \geq 0$

H.W 1.6: The two switches in the circuit seen in figure 1.33 operate synchronously. When switch 1 is in position a, switch 2 is closed. When switch 1 is in position (b), switch 2 is open. Switch 1 has been in position (a) for a long time. At $t = 0$ it moves instantaneously to position b. Find $v_c(t)$ for $t \geq 0$

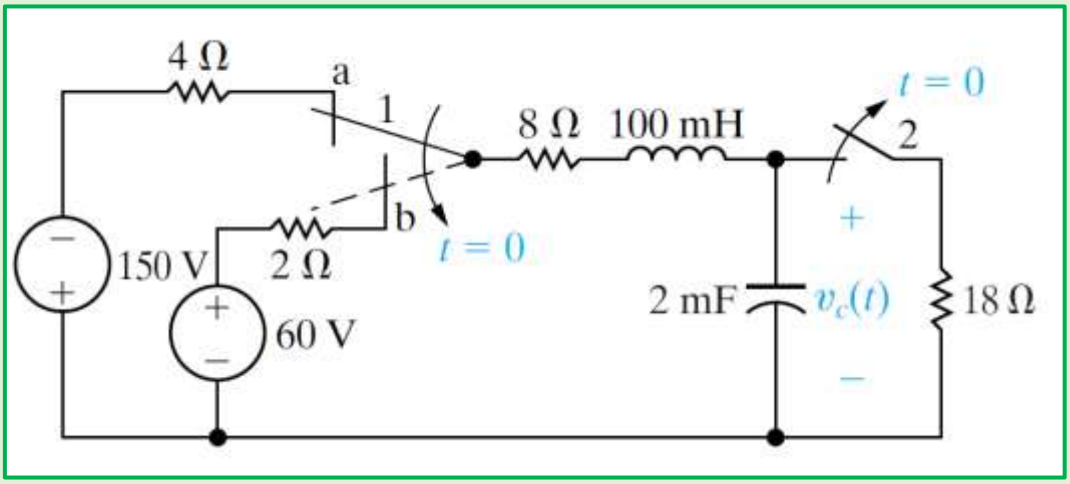

Figure 1.33

Ans: $v_c(t) = 60 - 150e^{-50t}\cos 50t - 200\, e^{-50t}\sin 50t\ V, \qquad t \geq 0$

HW 1.7: The two switches in the circuit seen in figure 1.34 operate synchronously. When switch 1 is in position (a), switch 2 is in position (d). When switch 1 moves to position (b), switch 2 moves to position (c). Switch 1 has been in position (a) for a long time. At $t = 0$, the switches move to their alternate positions. Find $v_o(t)$ for $t \geq 0$.

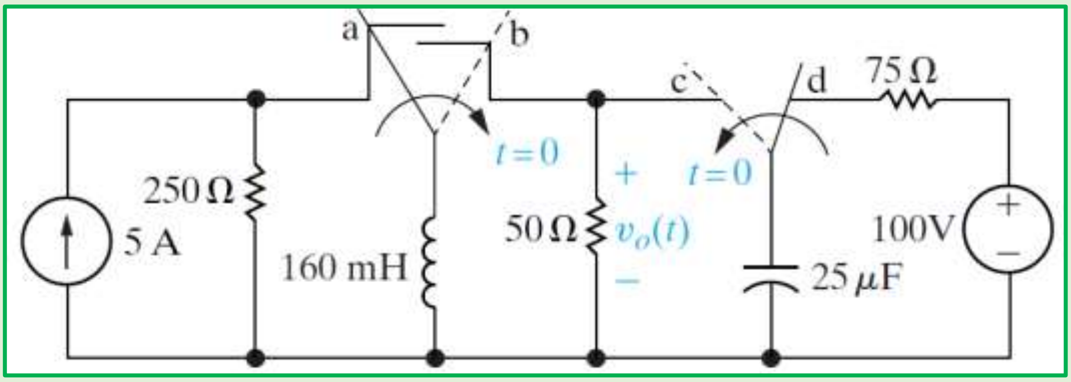

Figure 1.34

Ans: $\boldsymbol{v_o(t) = 100e^{-400t}\cos(300t) - 800e^{-400t}\sin(300t)\ V,\quad t \geq 0}$

HW 1.8: The switch in the circuit in figure 1.35 has been in the left position for a long time before moving to the right position at $t = 0$. Find

a) $i_L(t)$ for $t \geq 0$

b) $v_C(t)$ for $t \geq 0$

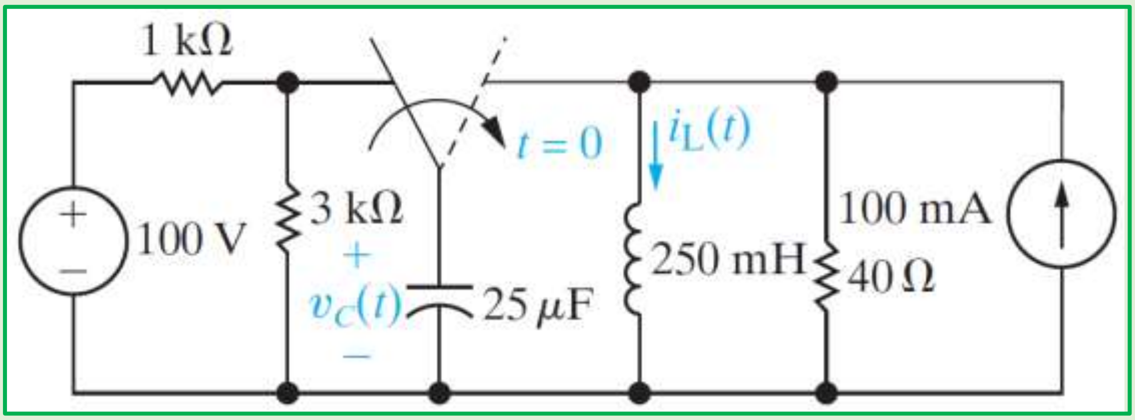

Figure 1.35

Ans: $\boldsymbol{i_L(t) = 0.1 + 0.5e^{-200t} - 0.5e^{-800t}\ \text{A},\ t \geq 0}$

$\boldsymbol{v_C(t) = -25e^{-200t} + 100e^{-800t}\ \text{V},\ t \geq 0}$

1.9 Applications

1) Photoflash Unit

- An electronic flash exploits the ability of the capacitor to oppose any abrupt change in voltage.
- Figure 1.36 consists essentially of a *high-voltage dc supply*, a *current-limiting large resistor R1*, and a *capacitor C* in parallel with the *flashlamp* of low resistance R2.

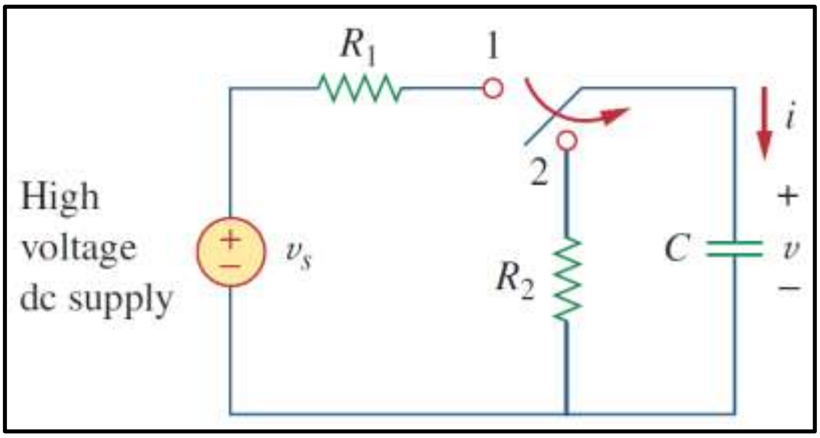

Figure 1.36

- When the switch is in position 1, the capacitor charges slowly due to the large time constant ($\tau = (R_1C)$) as shown in figure 1.37 (a).
- The capacitor voltage rises gradually from zero to Vs while its current decreases gradually from I_1=Vs/R_1 to zero.

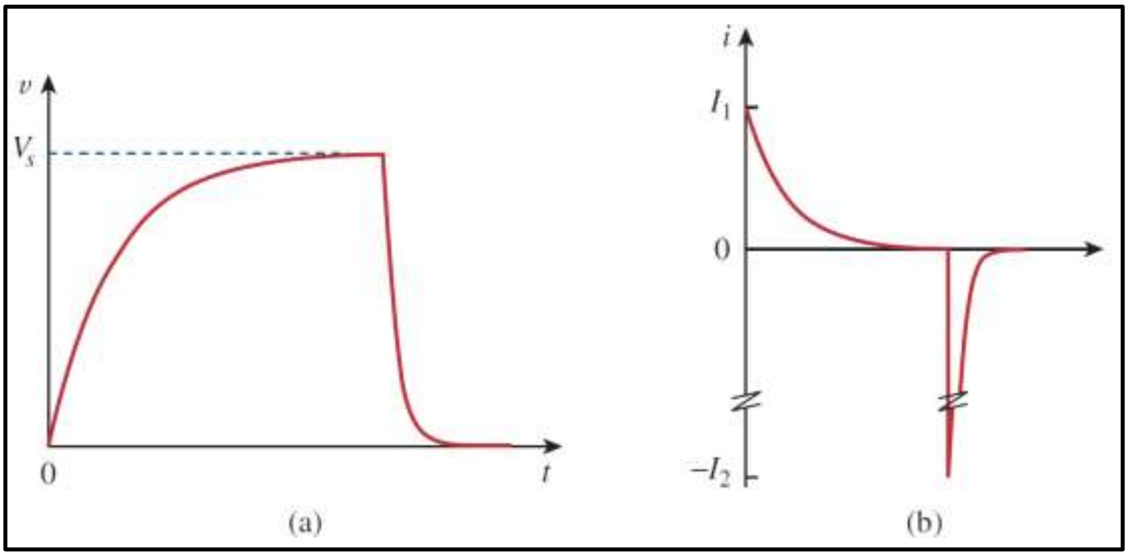

Figure 1.37

- The charging time is approximately five times the time constant, ($\tau = 5R_1C$), providing a short-duration, high current pulse.

- With the switch in position 2, the capacitor voltage is discharged. The low resistance R2 of the photo lamp permits a high discharge current with peak $I_2=Vs/R_1$ in a short duration as depicted in figure 1.37 (b), discharging takes place in approximately five times the time constant, ($\tau_{discharge} = 5R_2C$).

2) Relay Circuits

- A relay is essentially an electromagnetic device used to open or close a switch that controls another circuit.
- A typical relay circuit is shown in figure 1.38, the coil circuit is an RL circuit like that in figure 1.38 (b), where R and L are the resistance and inductance of the coil.
- When switch S_1 in figure 1.38 (a) is closed, the coil circuit is energized. The coil current gradually increases and produces a magnetic field. Eventually, the magnetic field is sufficiently strong to pull the movable contact in the other circuit and close switch S_2.
- At this point, the relay is said to be pulled in. The time interval t_d between the closure of switches S_1 and S_2 is called the relay delay time.
- Relays were used in the earliest digital circuits and are still used for switching high-power circuits.

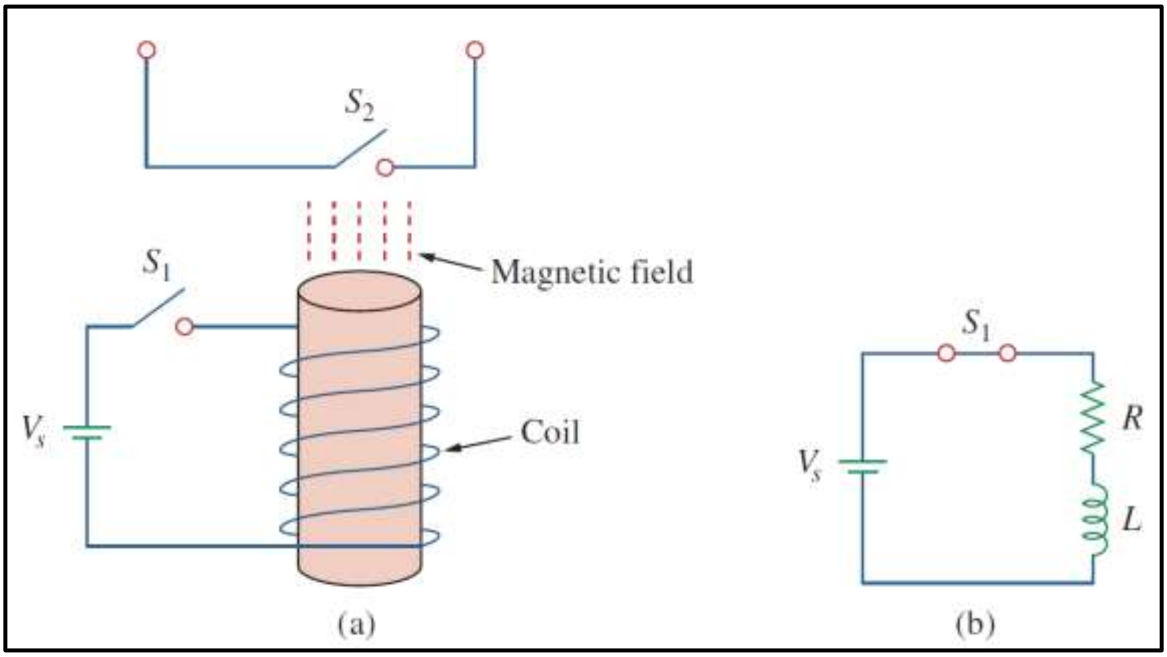

Figure 1.38

3) Automobile Ignition Circuit

- An automobile ignition system takes advantage of the ability of inductors to oppose a rapid change in current makes them useful for arc or spark generation.
- The gasoline engine of an automobile requires that the fuel-air mixture in each cylinder be ignited at proper times. This is achieved using a spark plug, which essentially consists of a pair of electrodes separated by an air gap.
- An inductor (the spark coil) L is used to create a large voltage (thousands of volts) between the electrodes (using the 12 V car battery), a spark is formed across the air gap, thereby igniting the fuel.
- When the ignition switch is closed, the current through the inductor increases gradually and reaches the final value of $i = Vs/R$ where $Vs = 12\ V$.
- The time taken for the inductor to charge is five times the time constant of the circuit $\tau = L/R,\ \ \tau_{charge} = 5L/R$
- Since at steady state, i is constant, $di/dt = 0$, and the inductor voltage v = 0.
- When the switch suddenly opens, a large voltage is developed across the inductor (due to the rapidly collapsing field) causing a spark or arc in the air gap. The spark continues until the energy stored in the inductor is dissipated in the spark discharge.
- The system is modeled by the circuit shown in figure 1.39. The 12-V source is due to the battery and alternator.

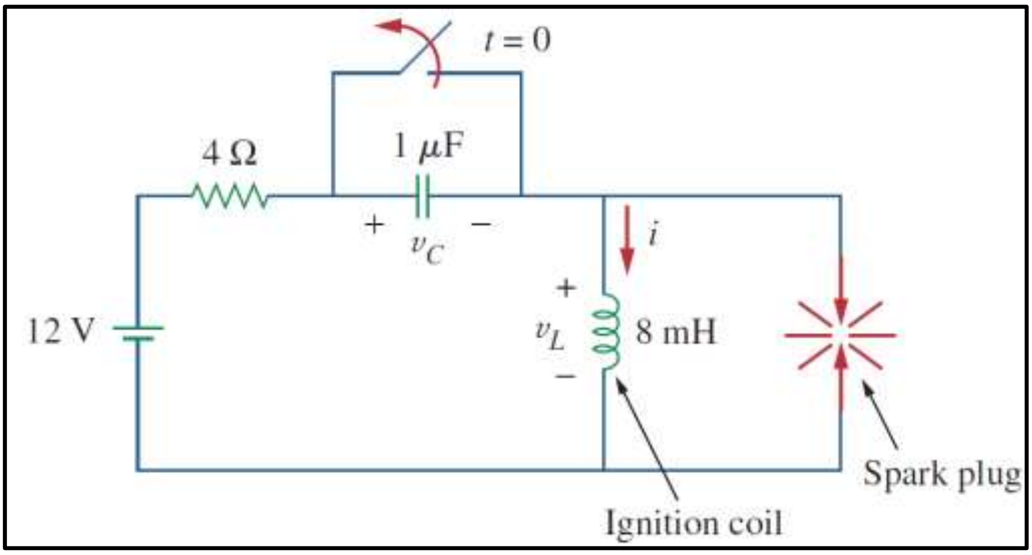

Figure 1.39

The 4 Ω resistor represents the resistance of the wiring. The ignition coil is modeled by the 8-mH inductor. The 1 μF capacitor (known as the condenser to auto-mechanics) is in parallel with the switch (known as the breaking points or electronic ignition).

H.W 1.9: An electronic flashgun has a current-limiting resistor 6 kΩ and 2000-μF electrolytic capacitor charged to 240 V. If the lamp resistance is 12 Ω find: (a) the peak charging current, (b) the time required for the capacitor to fully charge, (c) the peak discharging current, (d) the total energy stored in the capacitor, and (e) the average power dissipated by the lamp.

Ans: (a) 40 mA, (b) 1 minute, (c) 20 A, (d) 57.6 J, and (e) 480 watts

H.W 1.10: The coil of a certain relay is operated by a 12-V battery. If the coil has a resistance of 150 Ω and an inductance of 30 mH and the current needed to pull in is 50 mA, calculate the relay delay time.

(Ans: t_d = 0.1962 ms)

H.W 1.11: Assuming that the switch in figure 1.39 is closed prior to t = 0^- find the inductor voltage v_L for t > 0.

Ans: $v_L(t) - 268e^{-250t}\, sin(11,180t)\ V$