

Method of Separation of Variables

In this lecture we discuss the method of separation of variables. It involves a solution which breaks up into a product of two functions each of which contains only one of the variables. The method of separation of variables relies upon the assumption that a function of the form: $u(x, t) = F(x)G(t)$

The following examples explain this method.

Example 1: Apply the method of separation of variables to solve $u_x = 2u_t + u$

$$\text{with } u(x, 0) = 6e^{-3x}$$

Solution: Assume the solution is $u(x, t) = F(x)G(t)$

where F is a function of x alone and G is a function of t

Now we must find u_x and u_t : $u_x = F'G$ and $u_t = FG'$

Substituting in the given equation to get

$$F'G = 2FG' + FG \Leftrightarrow F'G - FG = 2FG'$$

$$(F' - F)G = 2FG' \Leftrightarrow \frac{F' - F}{2F} = \frac{G'}{G} = k$$

$$\frac{F' - F}{2F} = k \quad \text{and} \quad \frac{G'}{G} = k$$

$$F' - F = 2Fk \quad \text{and} \quad \ln G = kt + c_1$$

$$F' = 2Fk + F \quad \text{and} \quad G = e^{kt+c_1}$$

$$F' = (2k + 1)F \quad \text{and} \quad G = e^{kt}e^{c_1}$$

$$\frac{F'}{F} = 2k + 1 \quad \text{and} \quad G = ae^{kt}; \quad a = e^{c_1}$$

$$\ln F = (2k + 1)x + c_2 \quad \Leftrightarrow \quad F = e^{(2k+1)x+c_2}$$

$$F = be^{(2k+1)x} : b = e^{c_2}$$

$$\text{So } u(x, t) = be^{(2k+1)x} \times ae^{kt} = ce^{(2k+1)x+kt} : c = ab$$

$$u(x, 0) = 6e^{-3x} \Leftrightarrow 6e^{-3x} = ce^{(2k+1)x}$$

$$c = 6 \quad \text{and} \quad 2k + 1 = -3 \Leftrightarrow k = -2$$

$$\text{Then } u(x, t) = 6e^{-3x-2t}$$

Example 2: Solve the PDE by the method of separation of variables,

$$u_x = 2u_y + u \text{ with } u(x, 0) = 3e^{-5x} - 2e^{-3x}$$

Solution: $u = F(x)G(y) \Rightarrow u_x = F'G \text{ and } u_y = FG'$

$$F'G = 2FG' + FG \Rightarrow F'G = F(2G' + G)$$

$$\frac{F'}{F} = \frac{2G' + G}{G} \Rightarrow \frac{F'}{F} = \frac{2G'}{G} + 1 = k$$

$$\frac{F'}{F} = k \quad \text{and} \quad \frac{2G'}{G} + 1 = k \Rightarrow \frac{G'}{G} = \frac{k-1}{2}$$

$$\ln F = kx + c_1 \quad \text{and} \quad \ln G = \left(\frac{k-1}{2}\right)y + c_2$$

$$F = ae^{kx}; a = e^{c_1} \quad \text{and} \quad G = be^{\left(\frac{k-1}{2}\right)y}; b = e^{c_2}$$

$$u(x, y) = ae^{kx} \cdot be^{\left(\frac{k-1}{2}\right)y} = ce^{kx + \left(\frac{k-1}{2}\right)y}$$

$$u(x, 0) = 3e^{-5x} - 2e^{-3x} \Rightarrow 3e^{-5x} - 2e^{-3x} = c_1 e^{k_1 x} + c_2 e^{k_2 x}$$

$$\text{So, } c_1 = 3, k_1 = -5, c_2 = -2 \text{ and } k_2 = -3$$

$$u(x, y) = 3e^{-5x + \left(\frac{-5-1}{2}\right)y} - 2e^{-3x + \left(\frac{-3-1}{2}\right)y}$$

$$\text{Then } u(x, y) = 3e^{-5x-3y} - 2e^{-3x-2y}$$

Example 3: Solve $u_x - yu_y = 0$ with $u(0, y) = 2y^3 - 3y^2 + y$

Solution: $u = F(x)G(y) \Rightarrow u_x = F'G \text{ and } u_y = FG'$

$$F'G - yFG' = 0 \Rightarrow \frac{F'}{F} = \frac{yG'}{G} = k$$

$$\frac{F'}{F} = k \quad \text{and} \quad \frac{G'}{G} = \frac{k}{y} \Rightarrow F = ae^{kx} \quad \text{and} \quad G = by^k$$

$$u(x, y) = \sum_{n=1}^3 c_n y^{k_n} e^{k_n x} = c_1 y^{k_1} e^{k_1 x} + c_2 y^{k_2} e^{k_2 x} + c_3 y^{k_3} e^{k_3 x}$$

$$u(0, y) = 2y^3 - 3y^2 + y = c_1 y^{k_1} + c_2 y^{k_2} + c_3 y^{k_3}$$

$$\text{So, } c_1 = 2, k_1 = 3, c_2 = -3, k_2 = 2, c_3 = 1 \text{ and } k_3 = 1$$

$$\text{Then } u(x, y) = 2y^3 e^{3x} - 3y^2 e^{2x} + ye^x$$

Example 4: Solve $u_{xy} + u = 0$ with $u(0, y) = 2 \sinh(2y)$

Solution : $u = F(x)G(y) \Rightarrow u_x = F'G$ and $u_{xy} = F'G'$

$$F'G' + FG = 0 \Leftrightarrow \frac{F'}{F} = -\frac{G}{G'} = k$$

$$\frac{F'}{F} = k \text{ and } \frac{G'}{G} = -\frac{1}{k} \Leftrightarrow F = ae^{kx} \text{ and } G = be^{-\frac{y}{k}}$$

$$u(x, y) = ce^{kx - \frac{y}{k}}$$

$$u(0, y) = 2 \sinh(2y) \Leftrightarrow u(0, y) = e^{2y} - e^{-2y}$$

$$u(x, y) = c_1 e^{k_1 x - \frac{y}{k_1}} + c_2 e^{k_2 x - \frac{y}{k_2}}$$

$$e^{2y} - e^{-2y} = c_1 e^{-\frac{y}{k_1}} + c_2 e^{-\frac{y}{k_2}}$$

$$\text{So, } c_1 = 1, k_1 = -\frac{1}{2}, c_2 = -1 \text{ and } k_2 = \frac{1}{2}$$

$$u(x, y) = e^{-\frac{1}{2}x + y} - e^{\frac{1}{2}x - y} = 2 \sinh\left(y - \frac{x}{2}\right)$$

H.W: Apply the method of separation of variables to solve the PDE

1. $3u_x + 2u_y = 0$ with $u(x, 0) = 4e^{-x}$ *Ans.* $u(x, y) = 4e^{-x-1.5y}$
2. $2u_x - 3u_y = 0$ with $u(x, 0) = 5e^{3x}$ *Ans.* $u(x, y) = 5e^{3x+2y}$
3. $yu_x - xu_y = 0$ with $u(x, 0) = e^{-x^2}$ *Ans.* $u(x, y) = e^{-(x^2+y^2)}$
4. $u_{xy} - u = 0$ with $u(x, 0) = \cosh x$ *Ans.* $u(x, y) = \cosh(x + y)$