# المحاضرة التاسعة

## The exponential function and Trigonometric functions

## الدالة الأسية والدوال المثلثية

$$e^z = \exp(z) = e^{x+yi} = e^x[cosy + isiny]$$

**Ex.** express in form $x + yi$

1) $e^{1+\frac{\pi i}{6}}$

**Sol.** $e^1 . e^{\frac{\pi i}{6}} \quad \Rightarrow e^1 \left[cos\left(\frac{\pi}{6}\right) + isin\left(\frac{\pi}{6}\right)\right] \Rightarrow e\left[\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right] = \left[\frac{e\sqrt{3}}{2} + i\,\frac{e}{2}\right]$

2) $e^{2+3\pi i}$

**Sol.**$e^2 . e^{3\pi i} \Rightarrow e^2[cos3\pi + isin3\pi] \Rightarrow e^2[cos3\pi - 2\pi + isin3\pi - 2\pi]$

$= e^2[cos\pi + isin\pi] \Rightarrow e^2[-1+0] = -e^2$

3) $e^{\sqrt{3}-\frac{\pi i}{4}}$

**Sol**.$e^{\sqrt{3}} . e^{-\frac{\pi i}{4}} \Rightarrow e^{\sqrt{3}}\left[cos\left(\frac{-\pi}{4}\right) + isin\left(\frac{-\pi}{4}\right)\right] = e^{\sqrt{3}}\left[\cos\frac{\pi}{4} - isin\frac{\pi}{4}\right]$

$= e^{\sqrt{3}}\left[\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right] = \frac{e^{\sqrt{3}}}{\sqrt{2}} - \frac{e^{\sqrt{3}}}{\sqrt{2}}i$

**Proposition**: let $z = x + iy$ then

1. $e^{z_1+z_2} = e^{z_1} . e^{z_2}$
2. $e^{z_1-z_2} = \frac{e^{z_1}}{e^{z_2}}$
3. $|\boldsymbol{e^z}| = \boldsymbol{e^x}$ **Very important**
4. **arg**$(\boldsymbol{e^z}) = \boldsymbol{Imz + 2k\pi} \quad \boldsymbol{k} \in \boldsymbol{z}$

   $= y + 2k\pi \qquad k \in \mathbb{Z}$
5. $(e^z)^n = e^{nz} \quad n \in \mathbb{Z}$

6. $e^z \neq 0 \qquad \forall z \in \mathbb{C}$

7. $\frac{d}{dz} = [e^z] = e^z$

8. $e^z$ Is analytic function and entire function.

**Ex.** Prove that $\boldsymbol{e^{\bar{z}}} = \overline{\boldsymbol{e^z}}$

**Proof.** Let $z = x + yi$

$$\overline{(e^z)} = \left(\overline{e^{x+yi}}\right) \Rightarrow (\overline{e^x . e^{yi}}) = (e^{\bar{x}} . e^{\bar{y}}) \Rightarrow (e^x . e^{-y}) = \left(e^{x-yi}\right) = e^{\bar{z}}$$

**Ex.** solve the equations $e^z = 1 + \sqrt{3}i$

**Sol.** (1). $|e^z| = \left|1 + \sqrt{3}i\right| \Rightarrow e^x = 2 \quad \Rightarrow \ln e^x = \ln 2 \quad \Rightarrow \boldsymbol{x} = \ln 2$ **تم إيجاد قيمة**

(2). $y = \text{Arg}(e^z) + 2k\pi \qquad k \in \mathbb{Z}$

$y = \text{Arg}\left(1 + \sqrt{3}i\right) + 2k\pi \quad k \in \mathbb{Z}$

$y = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) + 2k\pi = \frac{\pi}{3} + 2k\pi$ **تم إيجاد قيمة**

$$\therefore z = x + yi \quad \Rightarrow z = \ln 2 + \left(\left(\frac{\pi}{3}\right) + 2k\pi\right)i \qquad k \in \mathbb{Z}$$

**Ex. solve the equations** $e^z = 1$

**Sol.** (1).$|e^z| = |1| \Rightarrow e^x = 1 \Rightarrow \ln e^x = \ln 1 \Rightarrow \boldsymbol{x = 0}$

(2). $y = \text{Arg}(e^z) + 2k\pi \Rightarrow y = \text{Arg}(1) + 2k\pi$

$$y = \tan^{-1}\left(\frac{0}{1}\right) + 2k\pi$$

$$y = 0 + 2k\pi$$

$$\therefore z = x + yi \quad \Rightarrow z = 0 + 2k\pi i \qquad k \in \mathbb{Z}$$

**Ex. solve the equations** $\boldsymbol{e^z = -4}$

**Sol.** (1). $|e^z| = |-4| \Rightarrow e^x = 4 \Rightarrow \ln e^x = \ln 4 \Rightarrow x = \ln 4$

(2). $y = \text{Arg}(e^z) + 2k\pi \qquad k \in \mathbb{Z}$

$$y = \text{Arg}(-4) + 2k\pi \implies y = \tan^{-1}\left(\frac{0}{-4}\right) + 2k\pi \implies y = \pi + 2k\pi$$

$$z = x + yi = \ln 4 + (\pi + 2k\pi)i$$

**H.W.** solve the equations $\boldsymbol{2e^{z} = 4i}$

**Ex. solve the equations** $e^{4z} = i$

Sol. (1). $|e^{4z}| = |i| \implies e^{4x} = 1 \implies \ln e^{4x} = \ln 1 \implies 4x = 0 \implies x = 0$

$$\because z = x + yi \implies 4z = 4x + 4yi$$

(2). $4y = \text{Arg}(e^{4z}) + 2k\pi \quad k \in \mathbb{Z}$

$$4y = \tan^{-1}\left(\frac{1}{0}\right) + 2k\pi \implies 4y = \frac{\pi}{2} + 2k\pi \implies y = \frac{\pi}{8} + 2k\pi$$

$$\therefore z = x + yi = 0 + \left(\frac{\pi}{8} + 2k\pi\right)i = \left(\frac{\pi}{8} + 2k\pi\right)i$$

**Ex. solve the equations** $e^{2z-1} = 1$

Sol. $\because 2z - 1 \implies 2x + 2yi - 1 \implies (2x - 1) + yi$

$$|e^{2z-1}| = |1| \implies e^{(2x-1)} = 1 \implies \ln e^{(2x-1)} = \ln 1 \implies 2x - 1 = 0 \implies \boldsymbol{x = \frac{1}{2}}$$

(2). $2y = \text{Arg}(e^{2z-1}) + 2k\pi \qquad k \in \mathbb{Z}$

$$2y = \tan^{-1}\left(\frac{0}{1}\right) + 2k\pi \implies 2y = 0 + 2k\pi \implies y = 2k\pi$$

$$\therefore z = x + yi = \frac{1}{2} + 2k\pi i \qquad k \in \mathbb{Z}$$

# Trigonometric functions

# الدوال المثلثية

$$\boldsymbol{\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \quad \cos z = \frac{e^{z} + e^{-iz}}{2}, \quad \tan z = \frac{\text{sinz}}{\text{cosz}} \implies \frac{\frac{e^{iz} - e^{-iz}}{2i}}{\frac{e^{z} + e^{-iz}}{2}} = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}}$$

**Ex.** Prove that $\frac{d}{dz}(sinz) = cosz$

**Proof.** $\frac{d}{dz}\left(\frac{e^{iz}-e^{-iz}}{2i}\right) \Longrightarrow \left(\frac{ie^{iz}+ie^{-iz}}{2i}\right) = \frac{i(e^{iz}+e^{-iz})}{2i} \Longrightarrow \frac{e^{iz}+e^{-iz}}{2} = cosz$

**Ex.** Prove that $\frac{d}{dz}(cosz) = -sinz$

**Proof.** $\frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right) \Longrightarrow \left(\frac{ie^{iz}-ie^{-iz}}{2}\right) = \frac{i(e^{iz}-e^{-iz})}{2} \times \frac{i}{i}$

$$\Longrightarrow \frac{i^2(e^{iz}-e^{-iz})}{2i} = -sinz$$

**Ex.** Prove that $\overline{cosz} = cos\bar{z}$

**Proof.**

$\overline{\left(\frac{e^{iz}+e^{-iz}}{2}\right)} = \left(\frac{\overline{e^{iz}+e^{-iz}}}{\bar{2}}\right)$ $\qquad$ $where\ \boldsymbol{e^{\bar{z}} = \overline{e^{z}}}$

$$\Longrightarrow \frac{\overline{e^{iz}}+\overline{e^{-iz}}}{2} = \frac{e^{i\bar{z}}+e^{-i\bar{z}}}{2} = cos\bar{z}$$

**Ex.** Evaluate.$\sin(i)$

**Sol.**$\sin(i) = \frac{e^{i(i)}-e^{-i(i)}}{2i} = \frac{e^{i^2}-e^{-i^2}}{2i} \quad \Longrightarrow \frac{e^{-1}-e}{2i} = \frac{\frac{1}{e}-e}{2i}$

**Ex.** Evaluate.$\cos(-i)$

**Sol**.$\cos(-i) = \frac{e^{i(-i)}+e^{-i(-i)}}{2} = \frac{e^{-i^2}-e^{i^2}}{2} \quad \Longrightarrow \frac{e+e^{-1}}{2} = \frac{e+\frac{1}{e}}{2}$

**Ex.** Evaluate.$\tan(2i)$

**Sol.** $\tan(2i) = \frac{e^{i(2i)}-e^{-i(2i)}}{i(e^{i(2i)}+e^{-i(2i)})} = \frac{e^{-2}-e^{2}}{i(e^{-2}+e^{2})} \quad \Longrightarrow \frac{\frac{1}{e^2}-e^2}{i\left(\frac{1}{e^2}+e^2\right)} = \frac{\frac{1-e^4}{e^2}}{i\left(\frac{1+e^4}{e^2}\right)} = \frac{1-e^4}{i(1+e^4)}$

**Ex. Prove that $(sinz)^2 + (cosz)^2 = 1$**

**Proof.** $\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2+\left(\frac{e^{z}+e^{-iz}}{2}\right)^2=1 \quad \Longrightarrow \frac{e^{2iz}-2+e^{-2iz}}{4i^2}+\frac{e^{2iz}+2+e^{-2iz}}{4}=1$

$$\frac{-e^{2iz}+2-e^{-2iz}+e^{2iz}+2+e^{-2iz}}{4}=\frac{4}{4}=1$$