Questions Form No. ⟨ 1 ⟩ Solve all the ODEs

[1] $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$

[2] $y'' - 3y' - 10y = 3e^{3x}$

[3] $x^2y'' + 5xy' + 4y = 1/x^2$

[4] $x\,y'' + y' + xy = 0$

Questions Form No. ⟨ 2 ⟩ Solve all the ODEs

[1] $x\dfrac{dy}{dx} + 3y = \dfrac{\cos x}{x^2}$

[2] $y'' + 4y' - 12y = 4e^{3x}$

[3] $x^2y'' + 5xy' + 4y = 1/x^2$

[4] $x\,y'' + y' + xy = 0$

Questions Form No. ⟨ 3 ⟩ Solve all the ODEs

[1] $x\dfrac{dy}{dx} + 3y = \dfrac{\sin x}{x^2}$

[2] $y'' - 3y' - 18y = 3e^{2x}$

[3] $x^2y'' + 5xy' + 4y = 1/x^2$

[4] $x\,y'' + y' + xy = 0$

Questions Form No. ⟨ 4 ⟩ Solve all the ODEs

[1] $x\dfrac{dy}{dx} + 3y = \dfrac{\cos x}{x^2}$

[2] $y'' + 3y' - 18y = 4e^{2x}$

[3] $x^2y'' + 5xy' + 4y = 1/x^2$

[4] $x\,y'' + y' + xy = 0$

Typical answers No. ⟨ 1 ⟩

[1] $x\dfrac{dy}{dx}+3y=\dfrac{\sin x}{x^2}$ ⇨ $\dfrac{dy}{dx}+\dfrac{3}{x}y=\dfrac{\sin x}{x^3}$ (**Linear ODE**)

$\mu(x)=e^{\int P(x)dx}=e^{\int \frac{3}{x}dx}$ ⇨ $\mu(x)=e^{3\ln x}$ ⇨ $\mu(x)=e^{\ln x^3}=x^3$

$y\,\mu(x)=\int \mu(x)\;Q(x)dx$ ⇨ $yx^3=\int x^3\times\dfrac{\sin x}{x^3}dx$

$yx^3=\int \sin x\;dx$ ⇨ $yx^3=-\cos x+C$ ⇨ $y=\dfrac{-\cos x+C}{x^3}$

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[2] $y''-3y'-10y=3e^{3x}$

$m^2-3m-10=0$ ⇨ $(m-5)(m+2)=0$ ⇨ $m=5,-2$

$y_h=c_1e^{5x}+c_2e^{-2x}$

$y_p=Ae^{3x}$ ⇨ $y_p'=3Ae^{3x}$ ⇨ $y_p''=9Ae^{3x}$

$9A-9A-10A=3$ ⇨ $A=-3/10$

So, $y_p=-3/10\,e^{3x}$. Then, $y=y_h+y_p=c_1e^{5x}+c_2e^{-2x}-3/10\,e^{2x}$

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[3] $x^2y''+5xy'+4y=1/x^2$ The given differential equation is Euler equation.

Put $x=e^t$. Then we get $\ddot{y}+(5-1)\dot{y}+4y=e^{-2t}$ OR $\ddot{y}+4\dot{y}+4y=e^{-2t}$

$m^2+4m+4=0$ ⇨ $m_1=m_2=-2$; $r=-2=m_1=m_2$

$y_h=(c_1t+c_2)e^{-2t}=c_1te^{-2t}+c_2e^{-2t}$,

$y_1=te^{-2t}$ and $y_2=e^{-2t}$ ⇨ $y_1'=-2te^{-2t}+e^{-2t}$ and $y_2'=-2e^{-2t}$

$v_1'y_1+v_2'y_2=0$ ⇨ $v_1'te^{-2t}+v_2'e^{-2t}=0$ ⇨ $v_1't+v_2'=0$ $\cdots(1)$

$v_1'y_1'+v_2'y_2'=R(x)$ ⇨ $v_1'(-2te^{-2t}+e^{-2t})-2v_2'e^{-2t}=e^{-2t}$

$v_1'(-2t+1)-2v_2'=1$ $\cdots(2)$

$2\times eq(1)\cdots$ $\underline{2v_1't+2v_2'=0}$ by summation.

$v_1'=1$ ⇨ $v_1=t$

Put $v_1'=1$ in $eq(1)$ to get $v_2'=-t$ ⇨ $v_2=-1/2\,t^2$

$y_p=v_1y_1+v_2y_2$ ⇨ $y_p=t^2e^{-2t}-1/2\,t^2e^{-2t}=1/2\,t^2e^{-2t}$

$y=y_h+y_p=(c_1t+c_2)e^{-2t}+1/2\,t^2e^{-2t}$

$y=\left(\dfrac{t^2}{2}+c_1t+c_2\right)e^{-2t}$ ⇨ $y=\left(\dfrac{(\ln x)^2}{2}+c_1\ln x+c_2\right)x^{-2}$

[4] $x\,y'' + y' + xy = 0$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots$$

$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$y'' = 2a_2 + 6a_3 x + 12a_4 x^2 + 20a_5 x^3 + 30a_6 x^4 + 42a_7 x^5 + \cdots$$

$$\boldsymbol{xy'' = 0 \quad + 2a_2 x + 6a_3 x^2 + 12a_4 x^3 + 20a_5 x^4 + 30a_6 x^5 + \cdots}$$

$$\boldsymbol{y' = a_1 \quad + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 \quad + 6a_6 x^5 + \cdots}$$

$$\boldsymbol{xy = 0 \quad + a_0 x \quad + a_1 x^2 \quad + a_2 x^3 \quad + a_3 x^4 \quad + a_4 x^5 \quad + \cdots}$$

$$\boxed{a_1 = 0}$$

$$2a_2 + 2a_2 + a_0 = 0 \quad ⇨ \quad \boxed{a_2 = -(1/4)\,a_0}$$

$$6a_3 + 3a_3 + a_1 = 0 \quad ⇨ \boxed{a_3 = 0}$$

$$12a_4 + 4a_4 + a_2 = 0 \quad ⇨ a_4 = -(1/16)a_2 \quad ⇨ \quad \boxed{a_4 = (1/64)a_0}$$

$$20a_5 + 5a_3 + a_3 = 0 \quad ⇨ \boxed{a_5 = 0}$$

$$y = a_0 - \left(\frac{1}{4}\right) a_0 x^2 + \left(\frac{1}{64}\right) a_0 x^4 + \cdots = a_0\left(1 - \frac{1}{4}x^2 + \frac{1}{64}x^4 + \cdots\right)$$

Typical answers No. 〈 2 〉

[1] $x\dfrac{dy}{dx} + 3y = \dfrac{\cos x}{x^2}$ ⇨ $\dfrac{dy}{dx} + \dfrac{3}{x}y = \dfrac{\cos x}{x^3}$ (**Linear ODE**)

$$\mu(x) = e^{\int P(x)dx} = e^{\int \frac{3}{x}dx} \quad ⇨ \quad \mu(x) = e^{3\ln x} \quad ⇨ \quad \mu(x) = e^{\ln x^3} = x^3$$

$$y\,\mu(x) = \int \mu(x)\ Q(x)dx \quad ⇨ \quad yx^3 = \int x^3 \times \frac{\cos x}{x^3}dx$$

$$yx^3 = \int \cos x\ dx \quad ⇨ \quad yx^3 = \sin x + C \quad ⇨ \quad y = \frac{\sin x + C}{x^3}$$

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[2] $y'' + 4y' - 12y = 4e^{3x}$

$$m^2 + 4m - 12 = 0 \ ⇨ \ (m-2)(m+6) = 0 \quad ⇨ \quad m = 2, -6$$

$$y_h = c_1 e^{2x} + c_2 e^{-6x}$$

$$y_p = Ae^{3x} \quad ⇨ \quad y_p' = 3Ae^{3x} \quad ⇨ \quad y_p'' = 9Ae^{3x}$$

$$9A + 12A - 12A = 4 \quad ⇨ \quad A = 4/9$$

So, $y_p = 4/9\,e^{3x}$. Then, $y = y_h + y_p = c_1 e^{2x} + c_2 e^{-6x} + 4/9\,e^{3x}$

Typical answers No. ⟨ 3 ⟩

[2] $y'' - 3y' - 18y = 3e^{2x}$

$m^2 - 3m - 18 = 0 \Rightarrow (m-6)(m+3) = 0 \Rightarrow m = 6, -3$

$y_h = c_1 e^{6x} + c_2 e^{-3x}$

$y_p = Ae^{2x} \Rightarrow y_p' = 2Ae^{2x} \Rightarrow y_p'' = 4Ae^{2x}$

$4A - 6A - 18A = 3 \Rightarrow A = -3/20$

So, $y_p = -3/20\, e^{2x}$. Then, $y = y_h + y_p = c_1 e^{6x} + c_2 e^{-3x} - 3/20\, e^{2x}$

Typical answers No. ⟨ 4 ⟩

[2] $y'' + 3y' - 18y = 4e^{2x}$

$m^2 + 3m - 18 = 0 \Rightarrow (m+6)(m-3) = 0 \Rightarrow m = -6,3$

$y_h = c_1 e^{-6x} + c_2 e^{3x}$

$y_p = Ae^{2x} \Rightarrow y_p' = 2Ae^{2x} \Rightarrow y_p'' = 4Ae^{2x}$

$4A + 6A - 18A = 4 \Rightarrow A = -1/2$

So, $y_p = -1/2\, e^{2x}$. Then, $y = y_h + y_p = c_1 e^{-6x} + c_2 e^{3x} - 1/2\, e^{2x}$