## Numerical methods for evaluating definite integrals

For definite integrals such as

$$\int_0^1 e^{-x^2} dx \quad \text{or} \quad \int_0^1 \sqrt{1-x^3}\, dx$$

we can't use the fundamental theorem of calculus to evaluate them since there are no elementary functions that are antiderivatives of $e^{-x^2}$ or $\sqrt{1-x^3}$. The best we can do is to use approximation methods for such integrals. We will see two methods that work reasonably well and yet are fairly simple.

### I- Trapezoidal rule

The trapezoidal rule is a numerical method that approximates the value of a definite integral. We consider the definite integral

$$\int_a^b f(x) dx$$

We assume that $f(x)$ is continuous on $[a, b]$ and we divide $[a, b]$ into $n$ subintervals of equal length

$$h = \frac{b-a}{n}$$

using the $n + 1$ points  P = n+1

$$x_0 = a,\ x_1 = a + h\ ,\ x_2 = a + 2h,\ \cdots\ ,\ x_{n-1} = a + (n-1)h,\ x_n = b$$

We can compute the value of $f(x)$ at these points.

$$y_0 = f(a), y_1 = f(a+h)\ ,\ y_2 = f(a+2h), \cdots, y_{n-1} = f(a+(n-1)h), y_n = f(b)$$

The trapezoidal rule can be computed as follows:

$$\int_a^b f(x) dx \cong \frac{h}{2}\left[\ y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\ \right]$$

f(a)  2f(a+h)  2f(a+2h)  f(b)

Example 1: Estimate $\int_0^1 e^{-x^2}dx$ by using trapezoidal rule for $n = 6$.

$$e^{-x^2} \rightarrow f(x)$$

$$\int_a^b f(x)dx \cong \frac{h}{2}\left[\; \underbrace{y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + 2y_5 + y_6}_{P=7} \;\right]$$

$P = n+1 = 7$

$$\underline{h} = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

$f(0)$ $x_0$

$x_1 = f(a+h) = y_1$ $x_1$

$x_2 = f(a+2h)$ $x_2$

$x_3$

⋮ $x_4$

$y_5 = f(a+5h)$ $x_5$

$x_6 = f(1)$ $x_6$

| $x_n$ | $y_n$ | Factors | Product |
|---|---|---|---|
| 0 | $y_0 = 1$ | 1 | 1 |
| $0 + 1/6$ | $y_1 = 0.9726$ | 2 | 1.9452 |
| $0 + 2/6$ | $y_2 = 0.8948$ | 2 | 1.7896 |
| $0 + 3/6$ | $y_3 = 0.7788$ | 2 | 1.5576 |
| $0 + 4/6$ | $y_4 = 0.6412$ | 2 | 1.2824 |
| $0 + 5/6$ | $y_5 = 0.4994$ | 2 | 0.9988 |
| 1 | $y_6 = 0.3679$ | 1 | 0.3679 |
| **Sum** | | | **8.9415** |

$$\int_0^1 e^{-x^2}dx \cong \frac{1/6}{2} \times 8.9415 = \frac{8.9415}{12} = 0.745125$$

Example 2: Estimate $\int_1^2 \sqrt{1+x^3}\,dx$ by using trapezoidal rule for $n = 5$.

$P = 6$

$$\int_a^b f(x)dx \cong \frac{h}{2}\left[\overbrace{y_0 + 2y_1 + 2y_2 + 2y_3 + 2y_4 + y_5}^{6}\right]$$

$$h = \frac{b-a}{n} = \frac{2-1}{5} = \frac{1}{5} = 0.2$$

$h = 0.2$

$y_0 = \sqrt{1+1^3} = \sqrt{2}$

$x_0 = 1$

$x_1 = 1 + h = 1 + 0.2$

$x_2 = 1 + 2h = 1 + 0.4$

$x_3 = 1 + 3h = 1 + 0.6$

$x_4 = 1 + 4h = 1 + 0.8$

$x_5 = 2$

$y_5 = \sqrt{9} = 3$

| $x_n$ | $y_n$ | Factors | Product |
|---|---|---|---|
| $x_0$ 1 | $y_0 = 1.4142$ | 1 | 1.4142 |
| $x_1$ 1.2 | $y_1 = 1.6517$ | 2 | 3.3034 |
| $x_2$ 1.4 | $y_2 = 1.9349$ | 2 | 3.8698 |
| $x_3$ 1.6 | $y_3 = 2.2574$ | 2 | 4.5148 |
| $x_4$ 1.8 | $y_4 = 2.6138$ | 2 | 5.2276 |
| $x_5$ 2 | $y_5 = 3$ | 1 | 3 |
| **Sum** | | | **21.3276** |

$$\int_1^2 \sqrt{1+x^3}\,dx \cong \frac{0.2}{2} \times 21.3276 = 2.13276$$

## II- Simpson's rule

If $n$ is an even then Simpson's rule is given by:

$$\int_a^b f(x)dx \cong \frac{h}{3}\left[\; y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n \;\right]$$

Example 3: Estimate $\int_0^1 \sqrt{1-x^2}\,dx$ by using Simpson's rule for $n = 4$.

$P=5$

$$\int_a^b f(x)dx \cong \frac{h}{3}\left[\; y_0 + 4y_1 + 2y_2 + 4y_3 + y_4 \;\right]$$

$$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4}$$

$y_0 = \sqrt{1-0^2} = 1$

$x_0 = a = 0$

$x_1 = a+h = 0+\frac{1}{4}$

$x_2 = a+2h = 0+\frac{2}{4}$

$x_3 = a+3h$

$x_4 = b = 1$

| $x_n$ | $y_n$ | Factors | Product |
|---|---|---|---|
| 0 | $y_0 = 1$ | 1 | 1 |
| $0+1/4$ | $y_1 = 0.9682$ | 4 | 3.8728 |
| $0+2/4$ | $y_2 = 0.866$ | 2 | 1.732 |
| $0+3/4$ | $y_3 = 0.6614$ | 4 | 2.6456 |
| 1 | $y_6 = 0$ | 1 | 0 |
| **Sum** | | | **9.2504** |

$$\int_0^1 \sqrt{1-x^2}\,dx \cong \frac{1/4}{3} \times 9.2504 = \frac{9.2504}{12} = 0.7709$$

$f(x)$

Example 4: Estimate $\int_0^1 e^{\sqrt{x}}dx$ by using Simpson's rule for $n = 6$. even

$$\int_a^b f(x)dx \cong \frac{h}{3}\left[\ y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6\ \right]$$

$P = 7$

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

$a = 0$
$b = 1$

$a = 0 = x_0$
$x_1 = a + h$
$x_2 = a + 2h$
$= 0 + 5 \times \frac{1}{6}$
$x_5 = a + 5h$
$x_6 = 1$

| $x_n$ | $y_n$ | Factors | Product |
|---|---|---|---|
| 0 | $y_0 = 1$ | 1 | 1 |
| $0 + 1/6$ | $y_1 = 1.5042$ | 4 | 6.0168 |
| $0 + 2/6$ | $y_2 = 1.7813$ | 2 | 3.5626 |
| $0 + 3/6$ | $y_3 = 2.0281$ | 4 | 8.1124 |
| $0 + 4/6$ | $y_4 = 2.2626$ | 2 | 4.5252 |
| $0 + 5/6$ | $y_5 = 2.4915$ | 4 | 9.966 |
| 1 | $y_6 = 2.7183$ | 1 | 2.7183 |
| **Sum** | | | **35.9013** |

$$\int_0^1 e^{\sqrt{x}}dx \cong \frac{1/6}{3} \times 35.9013 = \frac{35.9013}{18} = 1.9945$$

## Exercises

Estimate the following definite integral by using numerical methods

1. $\int_1^2 \frac{e^x}{x}dx$ for $n = 5$ T

2. $\int_0^1 \sqrt{1 + x^4}\,dx$ for $n = 6$ S