6 Directional Derivatives and Gradient Vectors:

We know that if $f(x, y)$ is differentiable, then the rate at which f changes with respect to t along a differentiable curve x=g(t), y=h(t) is

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right).\left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right)$$

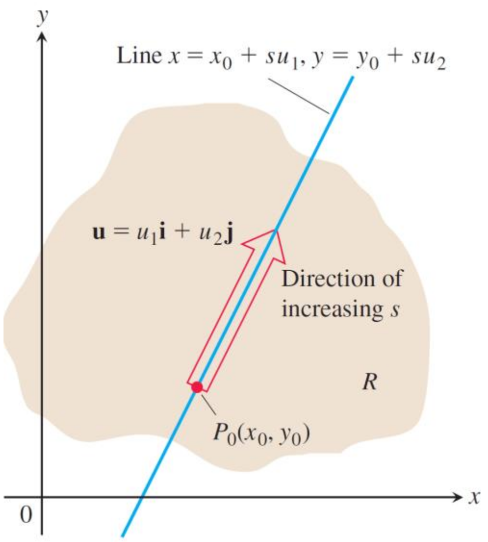
$$\frac{df}{dt} = \nabla f.\mathbf{v}$$

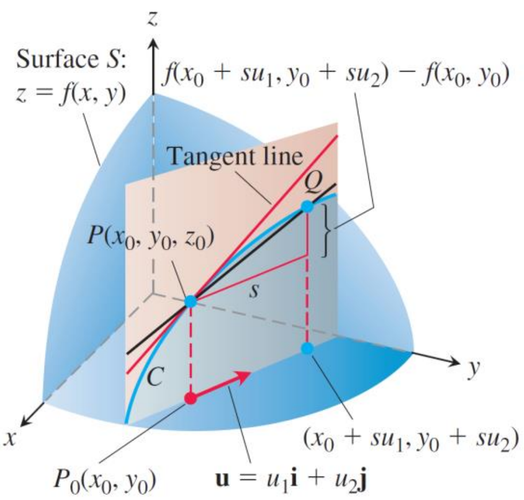
DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

The notation ∇f is read "grad f" as well as "gradient of f" and "del f." The symbol ∇ by itself is read "del." Another notation for the gradient is grad f.





Suppose that the function $f(x, y)$ is defined throughout a region R in the xy-plane, that $P_0(x_0, y_0)$ is a point in R, and that $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ is a unit vector. Then the equations

$$x = x_0 + su_1, \qquad y = y_0 + su_2, \qquad (2)$$

Parametrize the line through P_0 parallel to $\mathbf{u}$. If the parameter s measures arc length from P_0 in the direction of $\mathbf{u}$, we find the rate of change of f at P_0 in the direction of $\mathbf{u}$ by calculating df/ds at P_0

By the Chain Rule we find

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds} \qquad \text{Chain Rule for differentiable } f$$

$$= \left(\frac{\partial f}{\partial x}\right)_{P_0}u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0}u_2 \qquad \text{From Eqs. (2), } dx/ds = u_1 \text{ and } dy/ds = u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}. \qquad (3)$$

Equation (3) says that the derivative of a differentiable function f in the direction of $\mathbf{u}$ at P_0 is the dot product of $\mathbf{u}$ with the special vector, which we now define.

Using the gradient notation, we restate Equation (3) as a theorem.

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \qquad (4)$$

the dot product of the gradient ∇f at P_0 and $\mathbf{u}$. In brief, $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u}$.

EXAMPLE 2 Find the derivative of $f(x, y) = xe^y + \cos(xy)$ at the point $(2, 0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

Solution Recall that the direction of a vector $\mathbf{v}$ is the unit vector obtained by dividing $\mathbf{v}$ by its length:

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{\mathbf{v}}{5} = \frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}.$$

The partial derivatives of f are everywhere continuous and at $(2, 0)$ are given by

$$f_x(2, 0) = (e^y - y\sin(xy))_{(2,0)} = e^0 - 0 = 1$$

$$f_y(2, 0) = (xe^y - x\sin(xy))_{(2,0)} = 2e^0 - 2 \cdot 0 = 2.$$

The gradient of f at $(2, 0)$ is

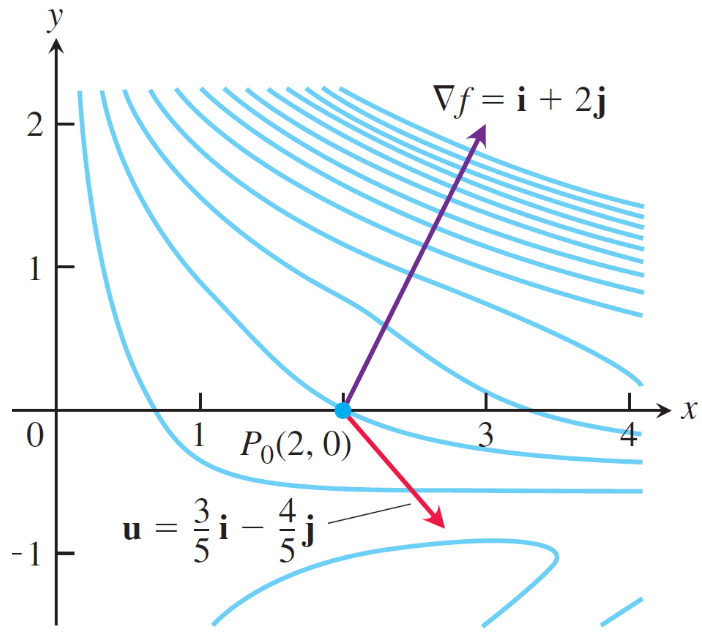
$$\nabla f|_{(2,0)} = f_x(2, 0)\mathbf{i} + f_y(2, 0)\mathbf{j} = \mathbf{i} + 2\mathbf{j}$$

The derivative of f at $(2, 0)$ in the direction of $\mathbf{u}$ is therefore

$$(D_{\mathbf{u}}f)_{(2,0)} = \nabla f|_{(2,0)} \cdot \mathbf{u} \qquad \text{Eq. (4) with the } (D_{\mathbf{u}}f)_{P_0} \text{ notation}$$

$$= (\mathbf{i} + 2\mathbf{j}) \cdot \left(\frac{3}{5}\mathbf{i} - \frac{4}{5}\mathbf{j}\right) = \frac{3}{5} - \frac{8}{5} = -1.$$

■



Evaluating the dot product in the brief version of Equation (4) gives

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||\mathbf{u}| \cos\theta = |\nabla f| \cos\theta,$$

where θ is the angle between the vectors $\mathbf{u}$ and ∇f, and reveals the following properties.

Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos\theta$

1. The function f increases most rapidly when $\cos\theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos(0) = |\nabla f|.$$
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f|\cos(\pi) = -|\nabla f|$.
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f|\cos(\pi/2) = |\nabla f| \cdot 0 = 0.$$

As we discuss later, these properties hold in three dimensions as well as two.

EXAMPLE 3 Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$

(a) increases most rapidly at the point (1, 1), and

(b) decreases most rapidly at (1, 1).

(c) What are the directions of zero change in f at (1, 1)?

Solution

(a) The function increases most rapidly in the direction of ∇f at (1, 1). The gradient there is

$$(\nabla f)_{(1,1)} = (x\mathbf{i} + y\mathbf{j})_{(1,1)} = \mathbf{i} + \mathbf{j}.$$

Its direction is

$$\mathbf{u} = \frac{\mathbf{i}+\mathbf{j}}{|\mathbf{i}+\mathbf{j}|} = \frac{\mathbf{i}+\mathbf{j}}{\sqrt{(1)^2+(1)^2}} = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}.$$

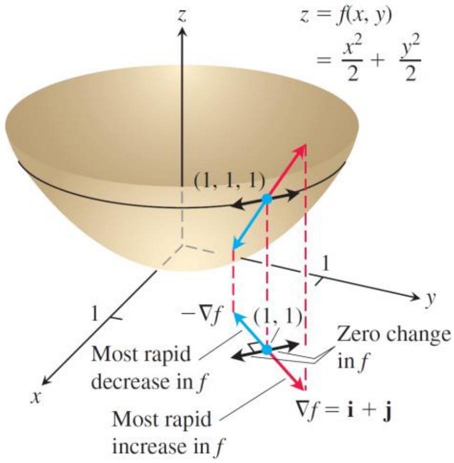
(b) The function decreases most rapidly in the direction of $-\nabla f$ at (1, 1), which is

$$-\mathbf{u} = -\frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

(c) The directions of zero change at (1, 1) are the directions orthogonal to ∇f:

$$\mathbf{n} = -\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j} \quad \text{and} \quad -\mathbf{n} = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j}.$$

See Figure 14.30. ■



If a differentiable function $f(x, y)$ has a constant value c along a smooth curve $\mathbf{r} = g(t)\mathbf{i} + h(t)\mathbf{j}$ (making the curve part of a level curve of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the equations

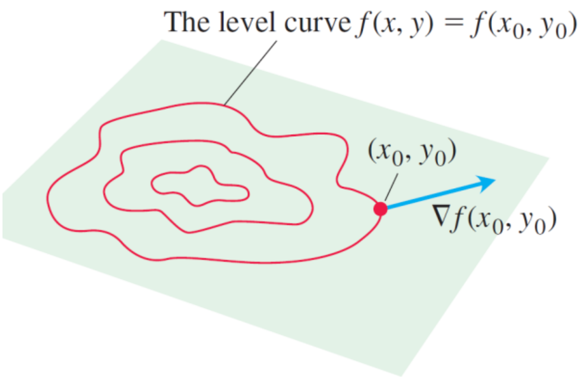
$$\frac{d}{dt} f(g(t), h(t)) = \frac{d}{dt}(c)$$

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \qquad \text{Chain Rule}$$

$$\underbrace{\left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right)}_{\nabla f} \cdot \underbrace{\left(\frac{dg}{dt}\mathbf{i} + \frac{dh}{dt}\mathbf{j}\right)}_{\frac{d\mathbf{r}}{dt}} = 0. \qquad (5)$$

Equation (5) says that ∇f is normal to the tangent vector $d\mathbf{r}/dt$, so it is normal to the curve.

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.31).



Equation (5) validates our observation that streams flow perpendicular to the contours in topographical maps (see Figure 14.26). Since the downflowing stream will reach its destination in the fastest way, it must flow in the direction of the negative gradient vectors from Property 2 for the directional derivative. Equation (5) tells us these directions are perpendicular to the level curves.

This observation also enables us to find equations for tangent lines to level curves. They are the lines normal to the gradients. The line through a point $P_0(x_0 , y_0)$ normal to a vector $\mathbf{N} = A\mathbf{i} + B\mathbf{j}$ has the equation

$$A(x - x_0) + B(y - y_0) = 0$$

If $\mathbf{N}$ is the gradient $(\nabla f)_{(x_o\,.y_o)} = f_x(x_o\,.y_o)\mathbf{i} + f_y(x_o\,.y_o)\mathbf{j}$, the equation gives the following formula.

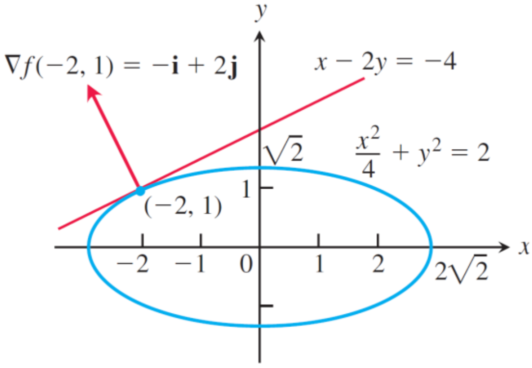
Tangent Line to a Level Curve

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \qquad (6)$$

EXAMPLE 4 Find an equation for the tangent to the ellipse

$$\frac{x^2}{4} + y^2 = 2$$

At the point (-2, 1).



Solution The ellipse is a level curve of the function

$$f(x, y) = \frac{x^2}{4} + y^2.$$

The gradient of f at $(-2, 1)$ is

$$\nabla f|_{(-2,1)} = \left(\frac{x}{2}\mathbf{i} + 2y\mathbf{j}\right)_{(-2,1)} = -\mathbf{i} + 2\mathbf{j}.$$

The tangent to the ellipse at $(-2, 1)$ is the line

$$(-1)(x + 2) + (2)(y - 1) = 0 \qquad \text{Eq. (6)}$$

$$x - 2y = -4.$$

Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f + g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f - g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$ — Scalar multipliers on left of gradients
5. *Quotient Rule:* $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

EXAMPLE 5 We illustrate two of the rules with

$$f(x, y) = x - y \qquad g(x, y) = 3y$$
$$\nabla f = \mathbf{i} - \mathbf{j} \qquad \nabla g = 3\mathbf{j}.$$

We have

1. $\nabla(f - g) = \nabla(x - 4y) = \mathbf{i} - 4\mathbf{j} = \nabla f - \nabla g$ — Rule 2
2. $\nabla(fg) = \nabla(3xy - 3y^2) = 3y\mathbf{i} + (3x - 6y)\mathbf{j}$

$= 3y(\mathbf{i} - \mathbf{j}) + 3y\mathbf{j} + (3x - 6y)\mathbf{j}$ — $g\nabla f$ plus terms . . .

$= 3y(\mathbf{i} - \mathbf{j}) + (3x - 3y)\mathbf{j}$ — simplified.

$= 3y(\mathbf{i} - \mathbf{j}) + (x - y)3\mathbf{j} = g\nabla f + f\nabla g$ — Rule 4

Functions of Three variables

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space, we have

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

and

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = \frac{\partial f}{\partial x}u_1 + \frac{\partial f}{\partial y}u_2 + \frac{\partial f}{\partial z}u_3.$$

The directional derivative can once again be written in the form

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f||u|\cos\theta = |\nabla f|\cos\theta,$$

EXAMPLE 6

(a) Find the derivative of $f(x, y, z) = x^3 - xy^2 - z$ at $P_0(1, 1, 0)$ in the direction of $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$.

(b) In what directions does f change most rapidly at P_0, and what are the rates of change in these directions?

Solution

(a) The direction of $\mathbf{v}$ is obtained by dividing $\mathbf{v}$ by its length:

$$|\mathbf{v}| = \sqrt{(2)^2 + (-3)^2 + (6)^2} = \sqrt{49} = 7$$

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}.$$

The partial derivatives of f at P_0 are

$$f_x = (3x^2 - y^2)_{(1,1,0)} = 2, \qquad f_y = -2xy|_{(1,1,0)} = -2, \qquad f_z = -1|_{(1,1,0)} = -1.$$

The gradient of f at P_0 is

$$\nabla f|_{(1,1,0)} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}.$$

The derivative of f at P_0 in the direction of $\mathbf{v}$ is therefore

$$(D_{\mathbf{u}}f)_{(1,1,0)} = \nabla f|_{(1,1,0)} \cdot \mathbf{u} = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} - \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)$$

$$= \frac{4}{7} + \frac{6}{7} - \frac{6}{7} = \frac{4}{7}.$$

(b) The function increases most rapidly in the direction of $\nabla f = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$ and decreases most rapidly in the direction of $-\nabla f$. The rates of change in the directions are, respectively,

$$|\nabla f| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = \sqrt{9} = 3 \qquad \text{and} \qquad -|\nabla f| = -3.$$

■

The Chain rule for paths

If r(t)=x(t)**i** + y(t)**j** + z(t)**k** is a smooth path C, and w=f(r(t)) is a scalar function evaluated along C, then according to the Chain Rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

The partial derivatives on the right-hand side of the above equation are evaluated along the curve $\mathbf{r}(t)$, and the derivatives of the intermediate variables are evaluated at t. If we express this equation using vector notation, we have

The Derivative Along a Path

$$\frac{d}{dt}f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t). \qquad (7)$$

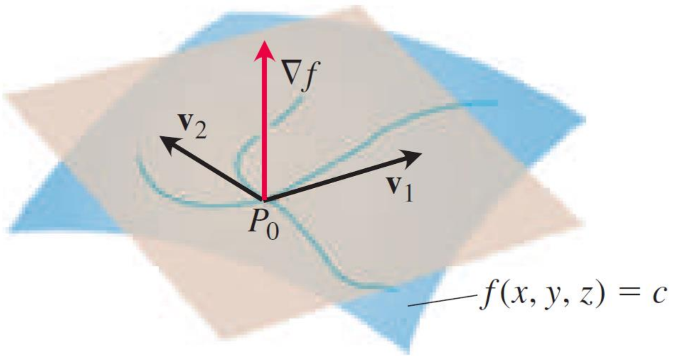
7 Tangent planes and Differentials

If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, we found in Equation (7) of the last section that

$$\frac{d}{dt} f(\mathbf{r}(t)) = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t).$$

Since f is constant along the curve $\mathbf{r}$, the derivative on the left-hand side of the equation is 0, so the gradient ∇f is orthogonal to the curve's velocity vector $\mathbf{r}'$.

Now let us restrict our attention to the curves that pass through P_0 See Figure below. All the velocity vectors at P_0 are orthogonal to ∇f at P_0, so the curves' tangent lines all lie in the plane through P_0 normal to ∇f. We now define this plane.



Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0 \quad (1)$$

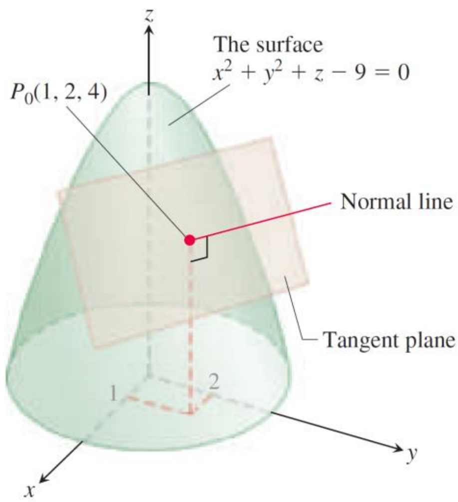
Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \quad y = y_0 + f_y(P_0)t, \quad z = z_0 + f_z(P_0)t \quad (2)$$

EXAMPLE 1 Find the tangent plane and normal line of the level surface

$$f(x, y, z) = x^2 + y^2 + z - 9 = 0 \quad \text{A circular paraboloid}$$

at the point $P_0(1, 2, 4)$.



Solution The surface is shown in Figure above.

The tangent plane is the plane through P_0 perpendicular to the gradient of f at P_0. The gradient is

$$\nabla f|_{P_0} = (2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k})_{(1,2,4)} = 2\mathbf{i} + 4\mathbf{j} + \mathbf{k}.$$

The tangent plane is therefore the plane

$$2(x - 1) + 4(y - 2) + (z - 4) = 0, \qquad \text{or} \qquad 2x + 4y + z = 14.$$

The line normal to the surface at P_0 is

$$x = 1 + 2t, \qquad y = 2 + 4t, \qquad z = 4 + t.$$

To find an equation for the plane tangent to a smooth surface $z = f(x, y)$ at a point $P_0(x_0, y_0, z_0)$ where $z_0 = f(x_0, y_0)$, we first observe that the equation $z = f(x, y)$ is equivalent to $f(x, y) - z = 0$. The surface $z = f(x, y)$ is therefore the zero level surface of the function $F(x, y, z) = f(x, y) - z$. The partial derivatives of F are

$$F_x = \frac{\partial}{\partial x}(f(x, y) - z) = f_x - 0 = f_x$$

$$F_y = \frac{\partial}{\partial y}(f(x, y) - z) = f_y - 0 = f_y$$

$$F_z = \frac{\partial}{\partial z}(f(x, y) - z) = 0 - 1 = -1.$$

The formula

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0$$

for the plane tangent to the level surface at P_0 therefore reduces to

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \qquad (3)$$

EXAMPLE 2 Find the plane tangent to the surface $z = x\cos y - ye^x$ at (0, 0, 0).

Solution We calculate the partial derivatives of $f(x, y) = x\cos y - ye^x$ and use Equation (3):

$$f_x(0, 0) = (\cos y - ye^x)_{(0,0)} = 1 - 0 \cdot 1 = 1$$
$$f_y(0, 0) = (-x\sin y - e^x)_{(0,0)} = 0 - 1 = -1.$$

The tangent plane is therefore

$$1 \cdot (x - 0) - 1 \cdot (y - 0) - (z - 0) = 0, \qquad \text{Eq. (3)}$$

or

$$x - y - z = 0.$$

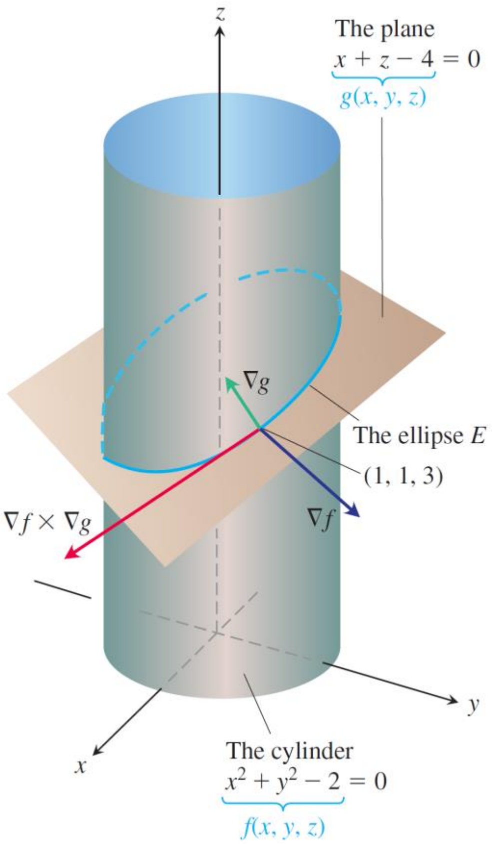
EXAMPLE 3 The surfaces

$$f(x, y, z) = x^2 + y^2 - 2 = 0 \qquad \text{A cylinder}$$

and

$$g(x, y, z) = x + z - 4 = 0 \qquad \text{A plane}$$

meet in an ellipse E (See the Figure below). Find parametric equations for the line tangent to E at the point $P_0(1, 1, 3)$.



Solution The tangent line is orthogonal to both ∇f and ∇g at P_0, and therefore parallel to $\mathbf{v} = \nabla f \times \nabla g$. The components of $\mathbf{v}$ and the coordinates of P_0 give us equations for the line. We have

$$\nabla f|_{(1,1,3)} = (2x\mathbf{i} + 2y\mathbf{j})_{(1,1,3)} = 2\mathbf{i} + 2\mathbf{j}$$

$$\nabla g|_{(1,1,3)} = (\mathbf{i} + \mathbf{k})_{(1,1,3)} = \mathbf{i} + \mathbf{k}$$

$$\mathbf{v} = (2\mathbf{i} + 2\mathbf{j}) \times (\mathbf{i} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 1 & 0 & 1 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} - 2\mathbf{k}.$$

The tangent line to the ellipse of intersection is

$$x = 1 + 2t, \qquad y = 1 - 2t, \qquad z = 3 - 2t. \qquad \blacksquare$$

H.W

(1) In Exercises 1–4, find the gradient of the function at the given point. Then sketch the gradient together with the level curve that passes through the point.

1. $f(x, y) = y - x, \quad (2, 1)$ **2.** $f(x, y) = \ln(x^2 + y^2), \quad (1, 1)$

3. $g(x, y) = xy^2, \quad (2, -1)$ **4.** $g(x, y) = \frac{x^2}{2} - \frac{y^2}{2}, \quad (\sqrt{2}, 1)$

(2) In Exercises 1–3, find the derivative of the function at P0 in the direction of u.

1. $f(x, y) = 2xy - 3y^2, \quad P_0(5, 5), \quad \mathbf{u} = 4\mathbf{i} + 3\mathbf{j}$
2. $f(x, y) = 2x^2 + y^2, \quad P_0(-1, 1), \quad \mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$
3. $g(x, y) = \frac{x - y}{xy + 2}, \quad P_0(1, -1), \quad \mathbf{u} = 12\mathbf{i} + 5\mathbf{j}$

(3) In Exercises 1–4, find the directions in which the functions increase and decrease most rapidly at P0. Then find the derivatives of the functions in these directions.

1. $f(x, y) = x^2 + xy + y^2, \quad P_0(-1, 1)$
2. $f(x, y) = x^2y + e^{xy} \sin y, \quad P_0(1, 0)$
3. $f(x, y, z) = (x/y) - yz, \quad P_0(4, 1, 1)$

(4) In Exercises 1–3, find equations for the
(a) tangent plane and
(b) normal line at the point P0 on the given surface..

1. $x^2 + y^2 + z^2 = 3, \quad P_0(1, 1, 1)$
2. $x^2 + y^2 - z^2 = 18, \quad P_0(3, 5, -4)$
3. $2z - x^2 = 0, \quad P_0(2, 0, 2)$

(5) In Exercises 1 and 2, find an equation for the plane that is tangent to the given surface at the given point.

1. $z = \ln(x^2 + y^2), \quad (1, 0, 0)$
2. $z = \sqrt{y - x}, \quad (1, 2, 1)$