Discrete Structures
1st Stage

Lec_1

Propositional Logic

Introduction

Logic is the basis of all mathematical reasoning. It has practical applications in areas of computer science as well as to many other fields of study. In mathematics, we must understand what makes up a correct mathematical argument, that is, a proof. Once we prove that a mathematical statement is true, we call it a theorem. A collection of theorems on a topic organize what we know about this topic. To learn a mathematical topic ,a person needs to actively construct mathematical arguments on this topic. Moreover, knowing the proof of a theorem often makes it possible to modify the result to fit new situations. Everyone knows that proofs are important throughout mathematics. The rules of logic give precise meaning to mathematical statements. These rules are used to distinguish between valid and invalid mathematical arguments. In this chapter, we will explain what makes up a correct mathematical argument and introduce tools to construct these arguments. These basic tools will help us to develop different proof methods that will enable us to prove many different types of results in the later chapters.

1.1 Basic Concepts in Logic

Our discussion begins with an introduction to the basic building blocks of logic viz., propositions.

Definition 1.1. A proposition is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

All the following declarative sentences are propositions.

1. New Delhi, is the capital of India.
2. $2 + 1 = 3$.
3. $2 + 1 = 2$.

Here propositions 1 and 2 are true, whereas 3 is false.
Some sentences that are not propositions are:

1. How are you?
2. Read this carefully.
3. $x + 1 = 2$.
4. $x + y = z$.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of the sentences 3 and 4 can be turned into a proposition if we assign values to the variables.

We use letters to denote propositional variables (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are $p, q, r, s, \ldots$. The truth value of a proposition is true, denoted by **T**, if it is a true proposition, and the truth value of a proposition is false, denoted by **F**, if it is a false proposition.

Definition 1.2. The area of logic that deals with propositions is called the propositional calculus or propositional logic.

Propositional calculus was first developed systematically by the greek philosopher Aristotle more than 2300 years ago.

Definition 1.3. Compound propositions are new propositions formed from existing propositions using logical operators.

Definition 1.4. Let p be a proposition. The negation of p, denoted by $\neg p$, is the statement "It is not the case that p." The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

The negation operator constructs a new proposition from a single existing proposition. We will now introduce the logical operators that are used to form new propositions from two or more existing propositions. These logical operators are also called connectives.

p	$\neg p$
T	F
F	T

Table 1.1: Negation

Definition 1.5. Let p and q be propositions. The conjunction of p and q, denoted by $p \wedge q$ is the proposition " p and q." The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 1.2: Conjunction(AND)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 1.3: **Disjunction(OR)**

Table 1.2 displays the truth table of $p \wedge q$. This table has a row for each of the four possible combinations of truth values of p and q. The four rows correspond to the pairs of truth values TT, TF, FT, and FF, where the first truth value in the pair is the truth value of p and the second truth value is the truth value of q. Note that in logic the word "but" sometimes is used instead of "and" in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining."

Definition 1.6. Let p and q be propositions. The **disjunction** of p and q, denoted by $p \lor q$, is the proposition "p or q." The disjunction $p \lor q$ is false when both p and q are false and is true otherwise.

Table 1.3 displays the truth table for $p \lor q$. The use of the connective "or" in a disjunction corresponds to one of the two ways the word "or" is used in English, namely, in an inclusive way. Thus, a disjunction is true when at least one of the two propositions in it is true. Sometimes, we use "or" in an exclusive sense. When the "exclusive or" is used to connect the propositions p and q, the proposition " p or q (but not both)" is obtained.

Definition 1.7. Let p and q be propositions. The **exclusive or** of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The truth table for the exclusive or of two propositions is displayed in Table 1.4.

Definition 1.8. Let p and q be propositions. The **conditional statement** $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Table 1.4: Exclusive or

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 1.5: Conditional Statement

In the conditional statement $p \to q, p$ is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence). The statement $p \to q$ is called a conditional statement because $p \to q$ asserts that q is true on the condition that p holds. A conditional statement is also called an implication. The truth table for the conditional statement $p \to q$ is shown in Table 1.5. Note that the statement $p \to q$ is true when both p and q are true and when p is false (no matter what truth value q has).
Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $p \to q$. A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, a pledge many politicians make when running for office is "If I am elected, then I will lower taxes." If the politician is elected but does not lowers taxes, then and then only the voters can say that the politician has broken the campaign pledge. This scenario corresponds to the case when p is true but q is false in $p \to q$.
You will encounter most if not all of the following ways to express this conditional statement :

"if p, then q "	" p implies q "
" if p , q "	" p only if q "
" p is sufficient for q "	"a sufficient condition for q is p "
" q if p "	" q whenever p "
" q when p "	" q is necessary for p "
" q unless ¬p "	" q follows from p "

"a necessary condition for p is q ".

Example 1.1. Let p be the statement "Ali learns discrete mathematics" and q the statement "Ali will find a good job." Express the statement $p \to q$ as a statement in English.
Solution: From the definition of conditional statements, we see that when p is the statement "Ali learns discrete mathematics" q is the statement "Ali will find a good job,"
$p \to q$ represents the statement "If Ali learns discrete mathematics, then he will find a good job."
There are many other ways to express this conditional statement in English. Among the most natural of these are:
"Ali will find a good job when Ali learns discrete mathematics."
"For Ali to get a good job, it is sufficient for him to learn discrete mathematics."
and "Ali will find a good job unless Ali does not learn discrete mathematics."
and so on.

1.1.1 Converse, Contrapositive, and Inverse

We can form some new conditional statements starting with a conditional statement $p \to q$. In particular, there are three related conditional statements that occur so often that they have special names.

Definition 1.9. The proposition $q \to p$ is called the converse of $p \to q$. The contrapositive of $p \to q$ is the proposition $\neg q \to \neg p$. The proposition $\neg p \to \neg q$ is called the inverse of $p \to q$.

From the truth table we can easily check that the truth values of $p \to q$ and $\neg q \to \neg p$ are same. This leads us to the next definition.

Definition 1.10. When two compound propositions always have the same truth value we call them **equivalent**.

The converse and the inverse of a conditional statement are also equivalent.

Example 1.2. What are the contrapositive, the converse, and the inverse of the conditional statement "The home team wins whenever it is raining?" Solution: Because " q whenever p " is one of the ways to express the conditional statement $p \to q$, the original statement can be rewritten as "If it is raining, then the

home team wins." Consequently, the contrapositive of this conditional statement is "If the home team does not win, then it is not raining." The converse is "If the home team wins, then it is raining." The inverse is "If it is not raining, then the home team does not win." Only the contrapositive is equivalent to the original statement. It can be easily verified by the truth table.

We now introduce another way to combine propositions that expresses that two propositions have the same truth value.

Definition 1.11. Let p and q be propositions. The **biconditional statement** $p \leftrightarrow q$ is the proposition " p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called bi-implications.

The truth table for $p \leftrightarrow q$ is shown in Table 1.6. There are some other common ways to express $p \leftrightarrow q$: " p is necessary and sufficient for q ", "if p then q, and conversely", " p iff q." The last way of expressing the biconditional statement $p \leftrightarrow q$ uses the abbreviation "iff" for "if and only if." Note that $p \leftrightarrow q$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$. We have now introduced four important

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 1.6: Biconditional Statement(X-NOR)

logical connectives–conjunctions, disjunctions, conditional statements, and biconditional statements–as well as negations. We can use these connectives to build up complicated compound propositions involving any number of propositional variables.We can use truth tables to determine the truth values of these compound propositions. We use a separate column to find the truth value of each compound expression that occurs in the compound proposition as it is built up. The truth values of the compound proposition for each combination of truth values of the propositional variables in it is found in the final column of the table.

Example 1.3. Construct the truth table of the compound proposition
$(p \vee \neg p) \rightarrow (p \wedge q)$

Solution: Because this truth table involves two propositional variables p and q, there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of p and q, respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \vee \neg q$, found in the fourth column. The fifth column gives the truth value of $p \wedge q$. Finally, the truth value of $(p \vee \neg q) \rightarrow (p \wedge q)$ is found in the last column. The resulting truth table is shown in Table 1.7.

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Table 1.7: The Truth table

1.1.2 Precedence of Logical Operators

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However, to reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q, namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q, namely $\neg(p \wedge q)$. Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \vee q \wedge r$ means $p \vee (q \wedge r)$ rather than $(p \vee q) \wedge r$. Because this rule may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear. Table 1.8 displays the precedence levels of the logical operators.

Operator	Precedence
$\neg$	1
$\wedge$	2
$\vee$	3
$\rightarrow$	4
$\leftrightarrow$	5

Table 1.8: Precedence of Logical Operators