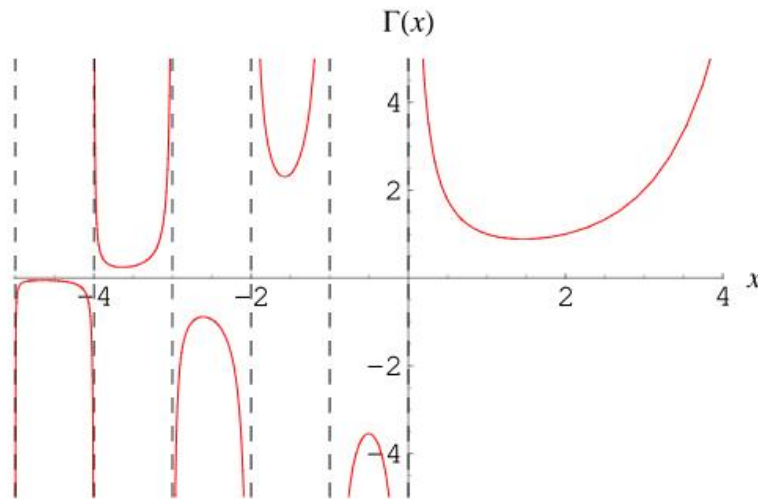


Gamma Function

The Gamma function is defined by the integral

$$\Gamma(n + 1) = \int_0^{\infty} t^n e^{-t} dt \quad ; n > 0$$



Plot of the Gamma Function

Example 1: $\int_0^{\infty} t^5 e^{-t} dt = \Gamma(5 + 1) = \Gamma(6)$

Gamma function satisfies the recursive properties:

1. $\Gamma(n) \neq 0$, for all n
2. $\Gamma(n + 1) = n!$ for all $n \in \mathbb{N}$
3. $\Gamma(n + 1) = n\Gamma(n)$ for all $n \neq 0, n \notin \mathbb{Z}^-$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
5. $\Gamma(p)\Gamma(1 - p) = \frac{\pi}{\sin p\pi}$; for all $0 < p < 1$

Example 2: Find 1. $\Gamma(5)$ 2. $\Gamma\left(\frac{3}{2}\right)$ 3. $\Gamma\left(\frac{5}{2}\right)$ 4. $\Gamma\left(-\frac{1}{2}\right)$ 5. $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

1. $\Gamma(5) = 4! = 4 \times 3 \times 2 \times 1 = 24$

2. $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}$

$$3. \Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{3}{4} \sqrt{\pi}$$

$$4. \Gamma(n+1) = n\Gamma(n) \quad \Leftrightarrow \quad \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2 \Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$5. \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1 - \frac{1}{4}\right) = \frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2}\pi$$

Example 3: Evaluate each of the following integrals

$$1. \int_0^{\infty} x\sqrt{x} e^{-x} dx = \int_0^{\infty} x^{\frac{3}{2}} e^{-x} dx = \Gamma\left(\frac{3}{2} + 1\right)$$

$$= \frac{3}{2} \times \frac{1}{2} \sqrt{\pi} = \frac{3}{4} \sqrt{\pi}$$

$$2. \int_0^{\infty} x^4 e^{-2x} dx$$

$$\text{Let } t = 2x \quad \Leftrightarrow \quad dt = 2dx \quad \Leftrightarrow \quad dx = \frac{dt}{2}$$

$$\int_0^{\infty} x^4 e^{-2x} dx = \int_0^{\infty} \left(\frac{t}{2}\right)^4 e^{-t} \frac{dt}{2} = \frac{1}{2^5} \int_0^{\infty} t^4 e^{-t} dx$$

$$= \frac{1}{32} \Gamma(5) = \frac{4!}{32} = \frac{24}{32} = \frac{3}{4}$$

Beta Function

Beta function is defined by the integral

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx; \quad n > 0, m > 0$$

Beta function satisfies the recursive properties:

1. The Beta function is symmetric, that is: $B(m, n) = B(n, m)$

$$2. B(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Example 4: Evaluate 1. $B(3,4)$ 2. $B\left(\frac{1}{2}, \frac{5}{2}\right)$

$$\begin{aligned} 1. B(3,4) &= \frac{\Gamma(3)\Gamma(4)}{\Gamma(7)} = \frac{2! \times 3!}{6!} \\ &= \frac{2 \times 3!}{6 \times 5 \times 4 \times 3!} = \frac{1}{60} \end{aligned}$$

$$\begin{aligned} 2. B\left(\frac{1}{2}, \frac{5}{2}\right) &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{5}{2}\right)} \\ &= \frac{\sqrt{\pi} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi}}{\Gamma(3)} = \frac{3\pi}{8} \end{aligned}$$

Example 4: Evaluate $\int_0^1 x^3(1-x)^4 dx$

$$\begin{aligned} \int_0^1 x^3(1-x)^4 dx &= B(4,5) = \frac{\Gamma(4)\Gamma(5)}{\Gamma(9)} = \frac{3! \times 4!}{8!} \\ &= \frac{6 \times 4!}{8 \times 7 \times 6 \times 5 \times 4!} = \frac{1}{280} \end{aligned}$$

Special Integrals

$$1. \int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \Gamma(p)\Gamma(1-p); \quad 0 < p < 1$$

$$2. \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta = \frac{1}{2} B(m, n)$$

Example 5: Evaluate 1. $\int_0^{\infty} \frac{dx}{\sqrt[3]{x^2}(1+x)}$ 2. $\int_0^{\pi/2} \sin^9 \theta \cos^5 \theta d\theta$ 3. $\int_0^{\pi/2} \sin^5 x dx$

$$1. \int_0^{\infty} \frac{dx}{\sqrt[3]{x^2}(1+x)} = \int_0^{\infty} \frac{x^{-2/3}}{1+x} dx = \int_0^{\infty} \frac{x^{(1/3)-1}}{1+x} dx = \frac{\pi}{\sin \frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

$$2. \quad 2m - 1 = 9 \Leftrightarrow m = 5 \quad \text{and} \quad 2n - 1 = 5 \Leftrightarrow n = 3$$

$$\int_0^{\pi/2} \sin^9 \theta \cos^5 \theta d\theta = \frac{1}{2} B(5,3) = \frac{\Gamma(5)\Gamma(3)}{2\Gamma(5+3)} = \frac{4! \times 2!}{2 \times 7!} = \frac{1}{210}$$

$$3. \quad 2m - 1 = 5 \Leftrightarrow m = 3 \quad \text{and} \quad 2n - 1 = 0 \Leftrightarrow n = \frac{1}{2}$$

$$\int_0^{\pi/2} \sin^5 x dx = \frac{1}{2} B\left(3, \frac{1}{2}\right) = \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)} = \frac{2! \sqrt{\pi}}{2 \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi}} = \frac{8}{15}$$

H. W. Evaluate each of the following integrals

$$1. \int_0^{\infty} x^5 e^{-3x} dx$$

$$2. \int_0^1 x^5 (1-x)^6 dx$$

$$3. \int_0^{\infty} \frac{1}{\sqrt[4]{x}(1+x)} dx$$

$$4. \int_0^{\pi/2} \cos^4 x dx$$

$$5. \int_0^{\pi/2} \sin^5 x \cos^4 x dx$$