Method of Separation of Variables

In this lecture we discuss the method of separation of variables. It involves a solution which breaks up into a product of two functions each of which contains only one of the variables. The method of separation of variables relies upon the assumption that a function of the form: $u(x,t) = F(x)G(t)$

The following examples explain this method.

Example 1: Apply the method of separation of variables to solve $u_x = 2u_t + u$ with $u(x,0) = 6e^{-3x}$

Solution: Assume the solution is $u(x,t) = F(x)G(t)$

where F is a function of x alone and G is a function of t

Now we must find u_x and u_t: $u_x = F'G$ and $u_t = FG'$

Substituting in the given equation to get

$F'G = 2FG' + FG \quad ⇨ \quad F'G - FG = 2FG'$

$(F' - F)G = 2FG' \quad ⇨ \quad \dfrac{F' - F}{2F} = \dfrac{G'}{G} = k$

$\dfrac{F' - F}{2F} = k$ and $\dfrac{G'}{G} = k$

$F' - F = 2Fk$ and $\ln G = kt + c_1$

$F' = 2Fk + F$ and $G = e^{kt+c_1}$

$F' = (2k+1)F$ and $G = e^{kt}e^{c_1}$

$\dfrac{F'}{F} = 2k + 1$ and $\boxed{G = ae^{kt}}$; $a = e^{c_1}$

$\ln F = (2k+1)x + c_2 \quad ⇨ \quad F = e^{(2k+1)x+c_2}$

$\boxed{F = be^{(2k+1)x}}$: $b = e^{c_2}$

So $u(x,t) = be^{(2k+1)x} \times ae^{kt} = ce^{(2k+1)x+kt}$: $c = ab$

$u(x,0) = 6e^{-3x} \quad ⇨ \quad 6e^{-3x} = ce^{(2k+1)x}$

$c = 6$ and $2k + 1 = -3 \quad ⇨ \quad k = -2$

Then $u(x,t) = 6e^{-3x-2t}$

Example 2: Solve $u_x = 2u_y + u$ with $u(x,0) = 3e^{-5x} - 2e^{-3x}$

Solution: $u = F(x)G(y) \Rightarrow u_x = F'G$ and $u_y = FG'$

$$F'G = 2FG' + FG \Rightarrow F'G = F(2G' + G)$$

$$\frac{F'}{F} = \frac{2G' + G}{G} \Rightarrow \frac{F'}{F} = \frac{2G'}{G} + 1 = k$$

$$\frac{F'}{F} = k \quad \text{and} \quad \frac{2G'}{G} + 1 = k \Rightarrow \frac{G'}{G} = \frac{k-1}{2}$$

$$\ln F = kx + c_1 \quad \text{and} \quad \ln G = \left(\frac{k-1}{2}\right)y + c_2$$

$$F = ae^{kx}; a = e^{c_1} \quad \text{and} \quad G = be^{\left(\frac{k-1}{2}\right)y}; \; b = e^{c_2}$$

$$u(x,y) = ae^{kx}.be^{\left(\frac{k-1}{2}\right)y} = ce^{kx+\left(\frac{k-1}{2}\right)y}$$

$$u(x,0) = 3e^{-5x} - 2e^{-3x} \Rightarrow 3e^{-5x} - 2e^{-3x} = c_1e^{k_1x} + c_2e^{k_2x}$$

So, $c_1 = 3$, $k_1 = -5$, $c_2 = -2$ and $k_2 = -3$

$$u(x,y) = 3e^{-5x+\left(\frac{-5-1}{2}\right)y} - 2e^{-3x+\left(\frac{-3-1}{2}\right)y}$$

Then $u(x,y) = 3e^{-5x-3y} - 2e^{-3x-2y}$

Example 3: Solve $u_{xy} + u = 0$ with $u(0,y) = 2\sinh(2y)$

Solution : $u = F(x)G(y) \Rightarrow u_x = F'G$ and $u_{xy} = F'G'$

$$F'G' + FG = 0 \Rightarrow \frac{F'}{F} = -\frac{G}{G'} = k$$

$$\frac{F'}{F} = k \quad \text{and} \quad \frac{G'}{G} = -\frac{1}{k} \Rightarrow \quad F = ae^{kx} \quad \text{and} \quad G = be^{-\frac{y}{k}}$$

$$u(x,y) = ce^{kx-\frac{y}{k}}$$

$$u(0,y) = 2\sinh(2y) \Rightarrow u(0,y) = e^{2y} - e^{-2y}$$

$$u(x,y) = c_1e^{k_1x-\frac{y}{k_1}} + c_2e^{k_2x-\frac{y}{k_2}}$$

$$e^{2y} - e^{-2y} = c_1e^{-\frac{y}{k_1}} + c_2e^{-\frac{y}{k_2}}$$

So, $c_1 = 1$, $k_1 = -0.5$, $c_2 = -1$ and $k_2 = 0.5$

$$u(x,y) = e^{-0.5x+y} - e^{0.5x-y} = 2\sinh(y - 0.5x)$$

Solution of Heat Equation

We want to solve the heat equation, $u_t = cu_{xx}$, $t > 0$, $0 \le x \le L$,
where $c > 0$ is a constant (the thermal conductivity of the material),
subject to the boundary: $u(0,t) = 0$, $u(L,t) = 0$ and the condition $u(x,0) = f(x)$.
The following examples explain how to do it.

Example 4: Apply the method of separation of variables to solve the heat equation
$u_t - 2u_{xx} = 0$ over $0 < x < 3$, $t > 0$ for the boundary conditions
$u(0,t) = u(3,t) = 0$ and the initial condition $u(x,0) = 5\sin 4\pi x$

Solution: Assume the solution is $u(x,t) = F(x)G(t)$

Then, $u_t = FG'$ and $u_{xx} = F''G$.

Substituting in $u_t - 2u_{xx} = 0$, to get

$FG' = 2F''G$

By separating the variables we get $\frac{F''}{F} = \frac{G'}{2G} = -\lambda^2$

$$\frac{F''}{F} = -\lambda^2 \quad \text{and} \quad \frac{G'}{2G} = -\lambda^2$$

$$F'' + \lambda^2 F = 0 \quad \text{and} \quad G' + 2G\lambda^2 = 0$$

$$F = A\sin\lambda x + B\cos\lambda x \quad \text{and} \quad G = Ce^{-2\lambda^2 t}$$

So $u(x,t) = (A\sin\lambda x + B\cos\lambda x) \times Ce^{-2\lambda^2 t}$

Or $u(x,t) = e^{-2\lambda^2 t}(D\sin\lambda x + E\cos\lambda x)$; $AC = D$, $BC = E$

$u(0,t) = 0$ since $\sin 0 = 0$ and $\cos 0 = 1$ this must imply that $E = 0$

$u(3,t) = 0 \Rightarrow De^{-2\lambda^2 t}\sin 3\lambda = 0 \Rightarrow e^{-2\lambda^2 t} \neq 0$

$D = 0 \Rightarrow u = 0$ (trivial solution)

The only sensible deduction is that $\sin 3\lambda = 0 \Rightarrow 3\lambda = n\pi \Rightarrow \lambda = n\pi/3$

Then $u(x,t) = D_n e^{-\frac{2n^2\pi^2}{9}t}\sin\frac{n\pi}{3}x$

$u(x,0) = 5\sin 4\pi x \Rightarrow 5\sin 4\pi x = D_n \sin\frac{n\pi}{3}x$

$4\pi = \frac{n\pi}{3} \Rightarrow n = 12$ so $D_{12} = 5$

Then $u(x,t) = 5e^{-\frac{288\pi^2}{9}t}\sin(4\pi x) = 5e^{-32\pi^2 t}\sin(4\pi x)$

Example 5: Solve by the method of separation of variables the heat equation

$$u_t = u_{xx} \ ; \ 0 < x < 1 \ , t > 0 \text{ with } u_x(0,t) = u_x(1,t) = 0$$

and the initial condition $u(x,0) = 3\cos 2\pi x$

Solution: Assume the solution is $u(x,t) = F(x)G(t)$

$u_t = FG'$ and $u_{xx} = F''G$

$FG' = F''G \Rightarrow \dfrac{F''}{F} = \dfrac{G'}{G} = -\lambda^2$

$\dfrac{F''}{F} = -\lambda^2$ and $\dfrac{G'}{G} = -\lambda^2$

$F'' + \lambda^2 F = 0$ and $G' + G\lambda^2 = 0$

$F = A\sin\lambda x + B\cos\lambda x$ and $G = Ce^{-\lambda^2 t}$

So $u(x,t) = (A\sin\lambda x + B\cos\lambda x) \times Ce^{-\lambda^2 t}$

Or $u(x,t) = e^{-\lambda^2 t}(D\sin\lambda x + E\cos\lambda x)$; $AC = D$, $BC = E$

$u_x(x,t) = e^{-\lambda^2 t}(D\lambda\cos\lambda x - E\lambda\sin\lambda x)$

$u_x(0,t) = 0 \Rightarrow D = 0$

$u_x(1,t) = 0 \Rightarrow -E\lambda e^{-\lambda^2 t}\sin\lambda = 0 \Rightarrow \lambda e^{-\lambda^2 t} \neq 0$

$E = 0 \Rightarrow u = 0$ (trivial solution)

$\sin\lambda = 0 \Rightarrow \lambda = n\pi$, $n = 1,2,3,\cdots$

Then $u(x,t) = E_n e^{-n^2\pi^2 t}\cos n\pi x$

$u(x,0) = 3\cos 2\pi x \Rightarrow 3\cos 2\pi x = E_n\cos n\pi x$

$2\pi = n\pi \Rightarrow n = 2$ so $E_2 = 3$

Then $u(x,t) = 3e^{-4\pi^2 t}\cos(2\pi x)$

Solution of Laplace's Equation

In this lecture we discuss the solution of Laplace's equation by using the method of separation of variables. The Laplace equation is written as: $u_{xx} + u_{yy} = 0$.

The following examples explain how to find the solution.

Example 6: Solve the Laplace's equation by the method of separation of variables over $0 \le x \le \pi, y > 0$ with $u(x,0) = u_x(0,y) = u_x(\pi,y) = 0$

Solution: Assume the solution is $u(x,y) = F(x)G(y)$ *then* $F''G + FG'' = 0$

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda^2$$

$$\frac{F''}{F} = -\lambda^2 \quad \Rightarrow \quad F'' + \lambda^2 F = 0$$

$$F = A \sin \lambda x + B \cos \lambda x$$

$$u_x(0,y) = u_x(\pi,y) = 0 \quad \Rightarrow \quad F'(0) = 0 \text{ and } F'(\pi) = 0$$

$$F' = \lambda A \cos \lambda x - \lambda B \sin \lambda x$$

$$F'(0) = 0 \Rightarrow 0 = \lambda A \cos(0) - \lambda B \sin(0) \Rightarrow A = 0$$

So $F = B \cos \lambda x$

$$F'(\pi) = 0 \quad \Rightarrow 0 = -\lambda B \sin(\lambda\pi)$$

Either $B = 0 \Rightarrow u = 0$ (trivial solution)

Or $\sin(\lambda\pi) = 0 \Rightarrow \lambda\pi = n\pi \Rightarrow \lambda = n$, $n = 1,2,3,\cdots$

So $F = B \cos nx$

Nwo $-\dfrac{G''}{G} = -\lambda^2 \Rightarrow G'' - n^2 G = 0$

$$G = C \sinh ny + D \cosh ny$$

$$u(x,0) = 0 \Rightarrow G(0) = 0 \Rightarrow 0 = C \sinh(0) + D \cosh(0) \Rightarrow D = 0$$

$$G = C \sinh ny$$

Then $u(x,y) = B \cos nx \times C \sinh ny$

$$u(x,y) = \sum_{n=1}^{\infty} a_n \cos nx \sinh ny$$

Example 7: Solve the Laplace's equation by the method of separation of variables over $0 \le x \le \pi, y > 0$ with $u(x,0) = u(0,y) = u_x(\pi,y) = 0$

Solution : Assume the solution is $u(x,y) = F(x)G(y)$ *then* $F''G + FG'' = 0$

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda^2$$

$$\frac{F''}{F} = -\lambda^2 \quad \Rightarrow \quad F'' + \lambda^2 F = 0$$

$$F = A\sin\lambda x + B\cos\lambda x$$

$$u(0,y) = 0 \quad \Rightarrow F(0) = 0$$

$$0 = A\sin(0) + B\cos(0) \quad \Rightarrow B = 0$$

$$\therefore \quad F = A\sin\lambda x$$

$$u_x(\pi,y) = 0 \quad \Rightarrow F'(\pi) = 0$$

$$F' = A\lambda\cos\lambda x \ \Rightarrow \ 0 = A\lambda\cos\lambda\pi \ \Rightarrow \ \cos\lambda\pi = 0$$

$$\lambda\pi = \frac{(2n-1)\pi}{2} \quad \Rightarrow \lambda = \frac{2n-1}{2} \qquad n = 1,2,3,\cdots$$

Nwo $-\dfrac{G''}{G} = -\lambda^2 \quad \Rightarrow \quad G'' - \lambda^2 G = 0$

$$G = C\sinh\lambda y + D\cosh\lambda y$$

$$u(x,0) = 0 \ \Rightarrow G(0) = 0 \ \Rightarrow 0 = C\sinh(0) + D\cosh(0) \ \Rightarrow D = 0$$

$$\therefore \ G = C\sinh\left(\frac{2n-1}{2}\right)y$$

Then $u(x,y) = A\sin\dfrac{(2n-1)x}{2} \times C\sinh\dfrac{(2n-1)y}{2} \qquad n = 1,2,3,\cdots$

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)x}{2}\sinh\frac{(2n-1)y}{2}$$

H.W: Apply the method of separation of variables to solve the PDE

1. $3u_x + 2u_y = 0$ with $u(x,0) = 4e^{-x}$ *Ans.* $u(x,y) = 4e^{-x-1.5y}$
2. $2u_x - 3u_y = 0$ with $u(x,0) = 5e^{3x}$ *Ans.* $u(x,y) = 5e^{3x+2y}$
3. $u_{xy} - u = 0$ with $u(x,0) = \cosh x$ *Ans.* $u(x,y) = \cosh(x+y)$
4. $u_t = u_{xx}$; $0 < x < 1$, $t > 0$ with $u(0,t) = u_x(1,t) = 0$ and $u(x,0) = 5\sin\frac{3\pi}{2}x$
5. $u_t = 3u_{xx}$ over $0 < x < \pi$, $t > 0$ for the boundary conditions $u(0,t) = u(\pi,t) = 0$ and $u(x,0) = 3\sin 2x - 6\sin 5x$
6. $u_{xx} + u_{yy} = 0$ with $u(x,0) = u_x(0,y) = u_x(\pi,y) = 0$
7. $u_{xx} + u_{yy} = 0$ with $u_y(x,0) = u(0,y) = u\left(\frac{\pi}{2},y\right) = 0$