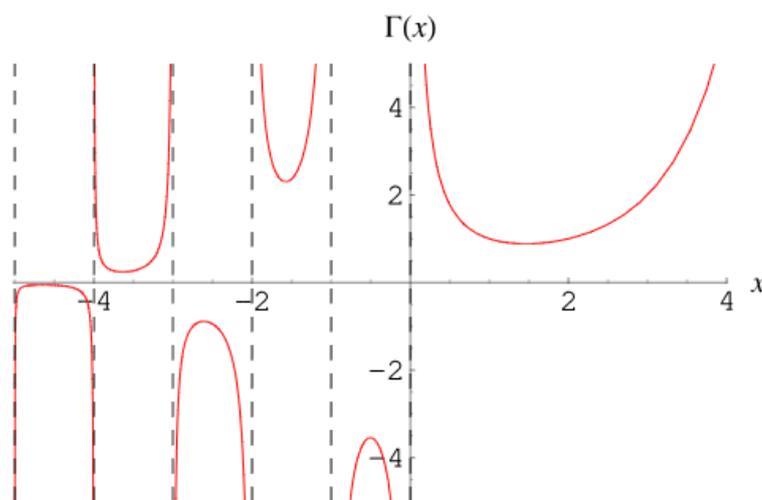


## Gamma Function

Gamma function is defined by the integral

$$\Gamma(n) = \int_0^{\infty} x^{n-1} e^{-x} dx \quad ; n > 0$$

Plot of the gamma function



Gamma function satisfies the recursive properties:

1.  $\Gamma(n + 1) = n\Gamma(n) \quad \forall n \neq 0, n \notin \mathbb{Z}^-$
2.  $\Gamma(n + 1) = n! \quad n \in \mathbb{N}$
3.  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
4.  $\int_0^{\infty} \frac{x^{p-1}}{1+x} dx = \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin p\pi} \quad ; 0 < p < 1$

**Example 1:** Find 1.  $\Gamma(5)$       2.  $\Gamma\left(\frac{3}{2}\right)$       3.  $\Gamma\left(\frac{5}{2}\right)$

4.  $\Gamma\left(-\frac{1}{2}\right)$       5.  $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$

1.  $\Gamma(5) = 4! = 4 \times 3 \times 2 \times 1 = 24$

2.  $\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2} + 1\right) = \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{1}{2} \sqrt{\pi}$

$$3. \Gamma\left(\frac{5}{2}\right) = \frac{3}{2} \times \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{3}{4} \sqrt{\pi}$$

$$4. \Gamma(n+1) = n\Gamma(n) \Leftrightarrow \Gamma(n) = \frac{\Gamma(n+1)}{n}$$

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2} + 1\right)}{-\frac{1}{2}} = -2 \Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}$$

$$5. \Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \Gamma\left(\frac{1}{4}\right)\Gamma\left(1 - \frac{1}{4}\right) = \frac{\pi}{\sin \frac{\pi}{4}} = \frac{\pi}{1/\sqrt{2}} = \sqrt{2} \pi$$

**Example 2:** Evaluate each of the following integrals

$$1. \int_0^{\infty} x\sqrt{x} e^{-x} dx = \int_0^{\infty} x^{\frac{3}{2}} e^{-x} dx = \Gamma\left(\frac{3}{2} + 1\right) \\ = \frac{3}{2} \times \frac{1}{2} \sqrt{\pi} = \frac{3}{4} \sqrt{\pi}$$

$$2. \int_0^{\infty} \frac{e^{-y^2}}{y^2} dy$$

$$\text{Let } x = y^2 \Leftrightarrow dx = 2y dy \Leftrightarrow dy = \frac{dx}{2\sqrt{x}}$$

$$\int_0^{\infty} \frac{e^{-y^2}}{y^2} dy = \int_0^{\infty} \frac{e^{-x}}{x} \frac{dx}{2\sqrt{x}} = \frac{1}{2} \int_0^{\infty} x^{-\frac{3}{2}} e^{-x} dx \\ = \frac{1}{2} \Gamma\left(-\frac{3}{2} + 1\right) = \frac{1}{2} \Gamma\left(-\frac{1}{2}\right) = -\sqrt{\pi}$$

$$3. \int_0^{\infty} t^{5/2} e^{-t/2} dt$$

$$\text{Let } t/2 = x \Leftrightarrow t = 2x \Leftrightarrow dt = 2dx \text{ and } t^{5/2} = 2^{5/2} x^{5/2} = 4\sqrt{2} x^{5/2}$$

$$\int_0^{\infty} t^{5/2} e^{-t/2} dt = \int_0^{\infty} 4\sqrt{2} x^{5/2} e^{-x} 2dx = 8\sqrt{2} \Gamma\left(\frac{5}{2} + 1\right) = 8\sqrt{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \sqrt{\pi} \\ = 15\sqrt{2}\pi$$

$$4. \int_0^{\infty} \frac{dx}{\sqrt[3]{x^2}(1+x)} = \int_0^{\infty} \frac{x^{-2/3}}{1+x} dx$$

$$= \int_0^{\infty} \frac{x^{(1/3)-1}}{1+x} dx = \frac{\pi}{\sin \frac{\pi}{3}} = \frac{2\pi}{\sqrt{3}}$$

**Example 3:** If  $A^2 \int_0^{\infty} e^{-2r/a_0} 4\pi r^2 dr = 1$ , show that  $A = \frac{1}{\sqrt{\pi a_0^3}}$

$$A^2 \int_0^{\infty} e^{-2r/a_0} 4\pi r^2 dr = 1 \quad \Leftrightarrow \quad 4\pi A^2 \int_0^{\infty} e^{-2r/a_0} r^2 dr = 1$$

To find the value of  $\int_0^{\infty} e^{-2r/a_0} r^2 dr$

$$\text{Let } 2r/a_0 = x \quad \Leftrightarrow \quad r = \frac{a_0}{2} x \quad \Leftrightarrow \quad dr = \frac{a_0}{2} dx \quad \text{and} \quad r^2 = \left(\frac{a_0}{2}\right)^2 x^2$$

$$\int_0^{\infty} e^{-2r/a_0} r^2 dr = \int_0^{\infty} e^{-x} \left(\frac{a_0}{2}\right)^2 x^2 \frac{a_0}{2} dx$$

$$= \left(\frac{a_0}{2}\right)^3 \int_0^{\infty} e^{-x} x^2 dx = \left(\frac{a_0}{2}\right)^3 \Gamma(3)$$

$$= \left(\frac{a_0}{2}\right)^3 \times 2 = \frac{a_0^3}{4}$$

$$\therefore 4\pi A^2 \frac{a_0^3}{4} = 1 \quad \Leftrightarrow \quad A^2 = \frac{1}{\pi a_0^3} \quad \Leftrightarrow \quad A = \frac{1}{\sqrt{\pi a_0^3}}$$

### Exercises

Evaluate each of the following integrals

$$1. \int_0^{\infty} x^6 e^{-3x} dx$$

$$2. \int_0^{\infty} 2t^7 e^{-t^2} dt$$

$$3. \int_0^{\infty} \frac{1}{\sqrt[4]{x}(1+x)} dx$$