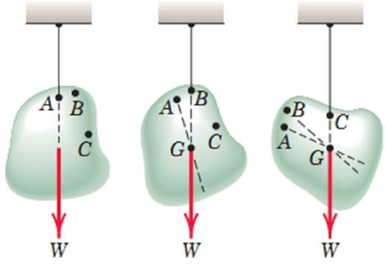
Distributed Forces

- The body will be in equilibrium under the action of the tension in the cord and the resultant W of the gravitational forces acting on all particles of the body. This resultant is clearly *collinear* with the cord.
- For all practical purposes these lines of action will be concurrent at a single point G, which is called the *center of gravity* of the body.



Determining the Center of Gravity

Apply the principle of moments to the parallel system of gravitational forces. The moment of the resultant gravitational force W about any axis equals the sum of the moments about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements of the body. The resultant of the gravitational forces acting on all elements is the weight of the body and is given by the sum $W = \int dW$.

The moment about y-axis of the elemental weight=xdW

The sum of these moments for all elements of the body about y-axis = $\int xdW$.

Moment of the sum =$\bar{x}W$

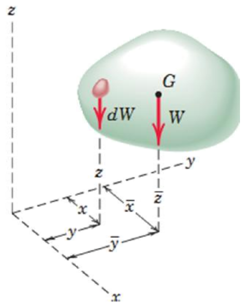
Moment of the sum must equal the sum of the moments, $\underbrace{\bar{x}W}_{\text{moment of sum}} = \underbrace{\int xdW}_{\text{sum of moments}} \rightarrow$

$$\bar{x} = \frac{\int xdW}{W}$$

Similar expressions for the other two coordinates, $\bar{y}, \bar{z}$ of the center of gravity G:

$$\left.\begin{array}{l} \bar{x} = \frac{\int xdW}{W} \\ \bar{y} = \frac{\int ydW}{W} \\ \bar{z} = \frac{\int zdW}{W} \end{array}\right\} \textit{center of gravity coordinate}$$

With the substitution of W= mg and dW= g dm, the expressions for the coordinates of the center of gravity become



$$\bar{x} = \frac{\int x\, g\, dm}{m\, g} \qquad \bar{y} = \frac{\int y\, g\, dm}{m\, g} \qquad \bar{z} = \frac{\int z\, g\, dm}{m\, g}$$

$$\left.\begin{array}{l} \bar{x} = \frac{\int x\, dm}{m} \\ \bar{y} = \frac{\int y\, dm}{m} \\ \bar{z} = \frac{\int z\, dm}{m} \end{array}\right\} \textit{center of mass coordinate}$$

Where, W: weight, m: mass, g: gravitational acceleration

If ρ, the density of a body is its mass per unit volume (V), then dm =ρdV

If ρ is not constant throughout the body

$$\left.\begin{array}{l} \bar{x} = \frac{\int x\, \rho dV}{\int \rho dV} \\ \bar{y} = \frac{\int y\, \rho dV}{\int \rho dV} \\ \bar{z} = \frac{\int z\, \rho dV}{\int \rho dV} \end{array}\right\} \textit{center of body coordinate}$$

Centroids of Lines, Areas, and Volumes

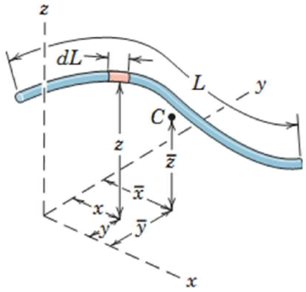
When the density of a body is uniform throughout, it will be a constant, then center of mass concise with geometrical center and termed *centroid*.

Centroid of Lines

L: length, A: cross-sectional area, ρ: density

If A and ρ are constant over the length of the rod, the coordinates of the center of mass also become the coordinates of the centroid C of the line segment

$$\left.\begin{aligned} \bar{x} &= \frac{\int x\, dL}{L} \\ \bar{y} &= \frac{\int y\, dL}{L} \\ \bar{z} &= \frac{\int z\, dL}{L} \end{aligned}\right\} \text{Centroid}$$



In general, the centroid C will not lie on the line. If the rod lies on a single plane, such as the x-y plane, only two coordinates need to be calculated.

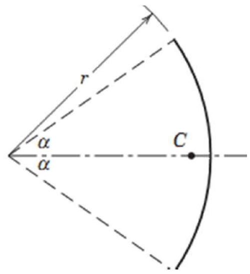
Remember,

$$_a^bL = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, .dx, \quad \text{if } y=f(x)$$

$$_c^dL = \int_c^d \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, .dy\,, \quad \text{if } x=g(y)$$

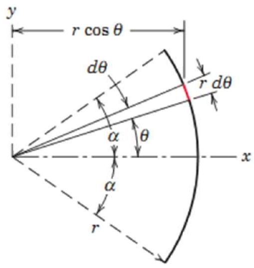
SAMPLE PROBLEM 5/1

Centroid of a *circular arc*. Locate the centroid of a circular arc as shown in the figure.



Solution.

Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length dL=rdθ expressed in polar coordinates, and the x-coordinate of the element is r cosθ



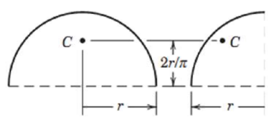
L= 2αr

$$\bar{x} = \frac{\int x\, dL}{L} \rightarrow = \frac{\int_{-\alpha}^{\alpha}(r\cos\theta)\ (rd\theta)}{2\alpha r} = \frac{r^2[\sin\theta]_{-\alpha}^{\alpha}}{2\alpha r} = \frac{r}{2\alpha}(\sin\alpha - \sin(-\alpha))$$

$$\bar{x} = \frac{r\sin\alpha}{\alpha}$$

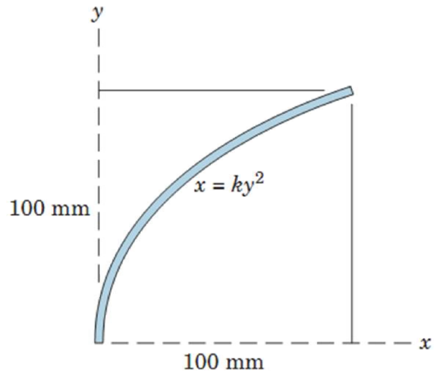
For a semicircular arc 2α=π, which gives $\bar{y} = \frac{r\sin\frac{\pi}{2}}{\frac{\pi}{2}} = \frac{2r}{\pi}$

For a quarter-circular arc 2α=π/2, which gives $\bar{x} = \bar{y} = \left(\frac{r\sin\frac{\pi}{4}}{\frac{\pi}{4}}\right)\cos\frac{\pi}{4} = \left(\frac{r\frac{1}{\sqrt{2}}}{\frac{\pi}{4}}\right)\frac{1}{\sqrt{2}} = \frac{2r}{\pi}$



Prob. 5/38

The homogeneous slender rod has a uniform cross section and is bent into the shape shown. Calculate the y-coordinate of the mass center of the rod.



Solution:

$$x = ky^2$$

$$@x = 100, y = 100 \rightarrow 100 = k100^2 \rightarrow k = \frac{1}{100}$$

$$dx/dy = 2ky$$

$$\bar{y} = \frac{\int y\,dL}{L}$$

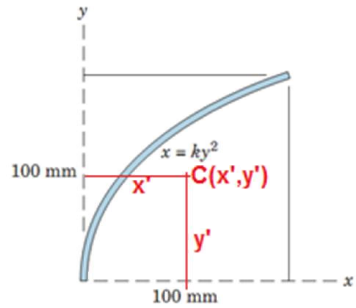
$$\int y\,dL = \int_0^{100} y\sqrt{1+(dx/dy)^2}.dy = \int_0^{100} y\sqrt{1+(2ky)^2}.dy = \int_0^{100} y\sqrt{1+4k^2y^2}.dy$$

$$\frac{8k^2}{8k^2}\int_0^{100} y\sqrt{1+4k^2y^2}.dy = \left[\frac{1}{8k^2}\frac{(1+4k^2y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{100} = \left[\frac{1}{8(0.01)^2}\frac{(1+4k^2y^2)^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{100}$$

$$= 8483\ mm^2$$

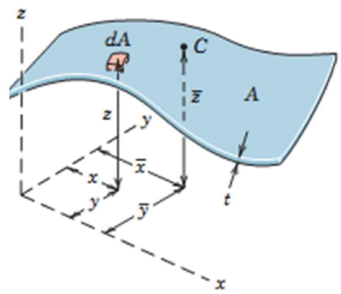
$$dL = \sqrt{1+(dx/dy)^2}.dy \rightarrow L = \int_0^{100}\sqrt{1+(dx/dy)^2}.dy = \int_0^{100}\sqrt{1+4k^2y^2}.dy = 148\ mm$$

$$\bar{y} = \frac{\int y\,dL}{L} = \frac{8483}{148} = 57.3\ mm$$



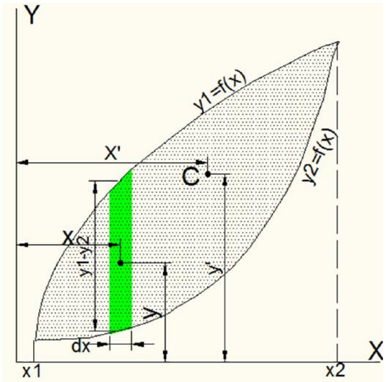
Centroids of Areas

When a body of density ρ has a small but constant thickness t, we can model it as a surface area A. The mass of an element becomes dm =ρ t dA. Again, if ρ and t are constant over the entire area, the coordinates of the center of mass of the body also become the coordinates of the centroid C of the surface area, and the coordinates may be written



$$\left.\begin{aligned}\bar{x} &= \frac{\overbrace{\int x\,dA}^{\text{first moments of area}}}{A}\\ \bar{y} &= \frac{\int y\,dA}{A}\\ \bar{z} &= \frac{\int z\,dA}{A}\end{aligned}\right\}\text{Centroid}$$

Choice of Element for Integration

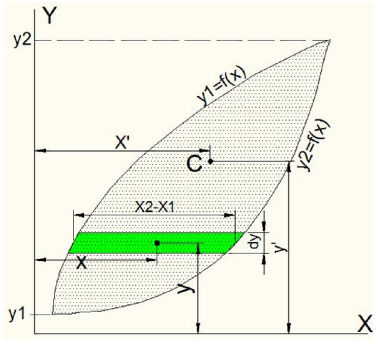


$$\bar{x} = \frac{\int x\,dA}{A} = \frac{\int_{x1}^{x2} x\,(y1 - y2)dx}{A}$$

$$A = \int_{x1}^{x2} (y1 - y2)dx$$

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{\int_{x1}^{x2} y(y1 - y2)dx}{A}$$

OR



$$\bar{x} = \frac{\int x\,dA}{A} = \frac{\int_{y1}^{y2} x\,(x2 - x1)dy}{A}$$

$$A = \int_{y1}^{y2} (x2 - x1)dy$$

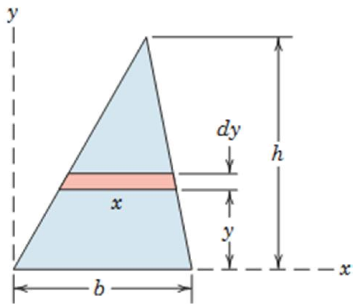
$$\bar{y} = \frac{\int y\,dA}{A} = \frac{\int_{y1}^{y2} y\,(x2 - x1)dy}{A}$$

SAMPLE PROBLEM 5/2

Centroid of a triangular area. Determine the distance $\bar{h}$ from the base of a triangle of altitude h to the centroid of its area.

Solution:

By similar triangles x/(h-y)=b/h→x=(b/h)(h-y)



$$\bar{h} = \bar{y} = \frac{\int y\,dA}{A}$$

$$M_x = \int y\,dA = \int_0^h y\,(xdy) = \int_0^h y\left(\frac{b}{h}\right)(h - y)dy$$

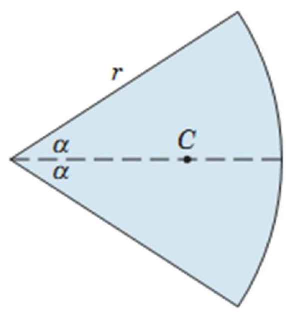
$$= \frac{b}{h}\int_0^h (hy - y^2)dy = \frac{b}{h}\left[\frac{hy^2}{2} - \frac{y^3}{3}\right]_0^h = \frac{bh^2}{6}$$

$$A = \int_0^h xdy = \int_0^h \left(\frac{b}{h}\right)(h - y)dy = \frac{bh}{2}$$

$$\bar{h} = \frac{\int y\,dA}{A} = \frac{\frac{bh^2}{6}}{\frac{bh}{2}} = \frac{h}{3}$$

SAMPLE PROBLEM 5/3

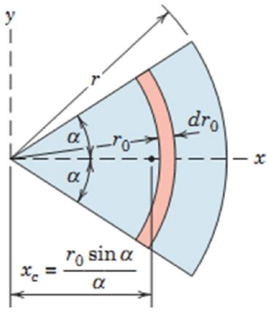
Centroid of the area of a *circular sector*. Locate the centroid of the area of a circular sector with respect to its vertex.



Solution I.

The x-axis is chosen as the axis of symmetry, and $\bar{y}$ is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0, so that its area is

$dA=2r_0\alpha dr_0$.



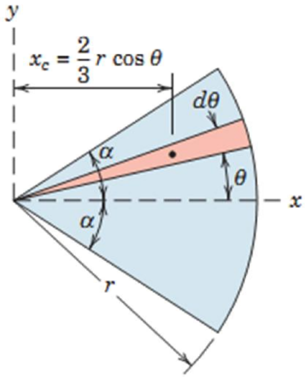
$$M_y = \int x\, dA = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right) (2r_0\alpha dr_0) = 2\sin\alpha \int_0^r {r_0}^2 dr_0$$

$$= 2\sin\alpha \left[\frac{{r_0}^3}{3}\right]_0^r = \frac{2\sin\alpha\, r^3}{3}$$

$$A = r^2\pi \frac{2\alpha}{2\pi} = r^2\alpha$$

$$\bar{x} = \frac{\int x\, dA}{A} = \frac{\frac{2\sin\alpha\, r^3}{3}}{r^2\alpha} = \frac{2r\sin\alpha}{3\alpha}$$

Solution II. Triangle of differential area



$$M_y = \int x\, dA = \int_{-\alpha}^{\alpha} \left(\frac{2}{3} r\cos\theta\right) \left(\frac{rd\theta}{2} r\right) = \frac{r^3}{3} \int_{-\alpha}^{\alpha} \cos\theta\, d\theta =$$

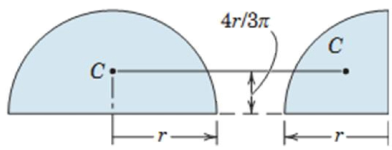
$$= \frac{r^3}{3} [\sin\theta]_{-\alpha}^{\alpha} = \frac{r^3}{3} [\sin\alpha - \sin -\alpha] = \frac{2}{3} r^3 \sin\alpha$$

$$A = r^2\pi \frac{2\alpha}{2\pi} = r^2\alpha$$

$$\bar{x} = \frac{\int x\, dA}{A} = \frac{\frac{2\sin\alpha\, r^3}{3}}{r^2\alpha} = \frac{2r\sin\alpha}{3\alpha}$$

For a *semicircular area* 2α=л, which gives $\bar{x} = \frac{2r\sin\alpha}{3\alpha} = \frac{2r\overbrace{\sin\frac{\pi}{2}}^{=1}}{3\frac{\pi}{2}} = \frac{4r}{3\pi}$

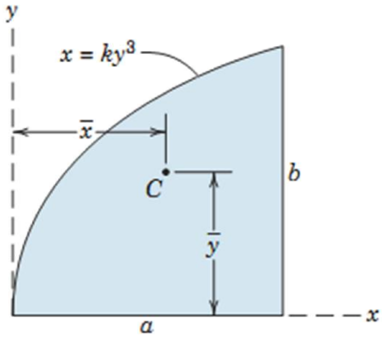
For a *quarter-circular area* 2α=л/2, which gives $\bar{x} = \bar{y} = \frac{4r}{3\pi}$



SAMPLE PROBLEM 5/4

Locate the centroid of the area under the curve $x=ky^3$ from x=0 to x=a.

Solution I. A vertical element

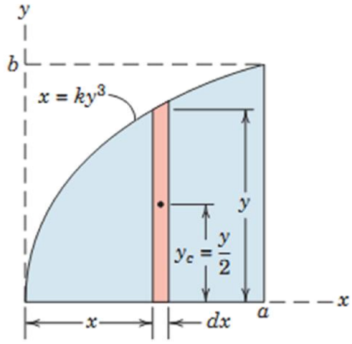


$$x = ky^3 \rightarrow a = kb^3 \rightarrow k = \frac{a}{b^3}$$

$$M_y = \int x\, dA = \int_0^{\alpha} x\, (ydx)$$

$$= \int_0^{\alpha} x \left(\sqrt[3]{\frac{x}{k}} dx\right) = \frac{1}{k^{\frac{1}{3}}} \int_0^{\alpha} x^{\frac{4}{3}}\, dx = \frac{1}{(\frac{a}{b^3})^{\frac{1}{3}}} \left[\frac{x^{\frac{7}{3}}}{\frac{7}{3}}\right]_0^{a} = \frac{3}{7} a^2 b$$

$$A=\int_0^a y\,dx=\int_0^a \sqrt[3]{\frac{x}{k}}\,dx=\frac{1}{k^{\frac{1}{3}}}\int_0^a x^{\frac{1}{3}}\,dx=\frac{1}{(\frac{a}{b^3})^{\frac{1}{3}}}\left[\frac{x^{\frac{4}{3}}}{\frac{4}{3}}\right]_0^a=\frac{3}{4}ab$$



$$\bar{x}=\frac{\int x\,dA}{A}=\frac{\frac{3}{7}a^2b}{\frac{3}{4}ab}=\frac{4}{7}a$$

$$M_x=\int \frac{y}{2}\,dA=\int_0^a \frac{y}{2}\,(ydx)=\int_0^a \frac{y^2}{2}\,dx=\frac{1}{2}\int_0^a [(\frac{x}{k})^{\frac{1}{3}}]^2dx$$

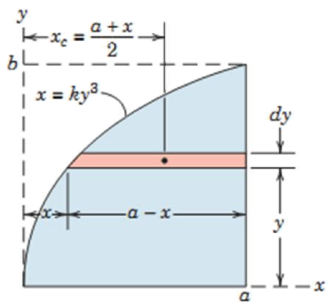
$$=\frac{1}{2(k)^{\frac{2}{3}}}\int_0^a x^{\frac{2}{3}}dx=\frac{1}{2(\frac{a}{b^3})^{\frac{2}{3}}}\left[\frac{x^{\frac{5}{3}}}{\frac{5}{3}}\right]_0^a=\frac{3}{10}ab^2$$

$$\bar{y}=\frac{\int \frac{y}{2}\,dA}{A}=\frac{\frac{3}{10}ab^2}{\frac{3}{4}ab}=\frac{2}{5}b$$

Solution II. The horizontal element of area

$$M_y=\int x\,dA=\int_0^b \left(\frac{a+x}{2}\right)[(a-x)dy]=\int_0^b \left(\frac{a+ky^3}{2}\right)[(a-ky^3)dy]=\frac{3}{7}a^2b$$

$$A=\int_0^b (a-x)dy=\int_0^b (a-ky^3)dy=\frac{3}{4}ab$$



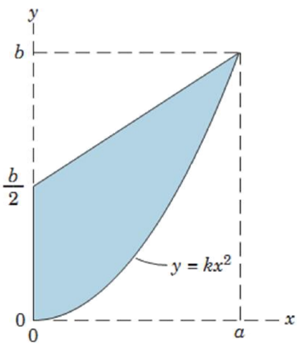
$$\bar{x}=\frac{\int x\,dA}{A}=\frac{\frac{3}{7}a^2b}{\frac{3}{4}ab}=\frac{4}{7}a$$

$$M_x=\int y\,dA=\int_0^b y\,[(a-x)dy]=\int_0^b y\,[(a-ky^3)dy]=\frac{3}{10}ab^2$$

$$\bar{y}=\frac{\int \frac{y}{2}\,dA}{A}=\frac{\frac{3}{10}ab^2}{\frac{3}{4}ab}=\frac{2}{5}b$$

Prob.5/18

Determine the coordinates of the centroid of the shaded area.



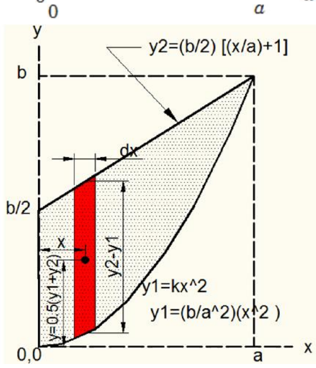
Solution:

$$y_1=k\,x^2\rightarrow b=k\,a^2\rightarrow k=\frac{b}{a^2}$$

$$y_1=\frac{b}{a^2}x^2$$

$$m=\frac{\Delta y}{\Delta x}=\frac{b-0.5b}{a-0}=\frac{b}{2a}$$

$$m=\frac{b}{2a}=\frac{y-0.5b}{x-0}\rightarrow y_2=\frac{b}{2}\left(\frac{x}{a}+1\right)$$



$$A=\int_0^a (y2-y1)dx=\int_0^a \left(\left[\frac{b}{2}\left(\frac{x}{a}+1\right)\right]-\left[\frac{b}{a^2}\,x^2\right]\right)dx$$

$$=\left[\left[\frac{b}{2}\left(\frac{x^2}{2a}+x\right)\right]-\left[\frac{b}{a^2}\,\frac{x^3}{3}\right]\right]_0^a=\frac{5}{12}ba$$

$$M_y=\int x\,dA=\int_0^a x(y2-y1)dx=\int_0^a x\left(\left[\frac{b}{2}\left(\frac{x}{a}+1\right)\right]-\left[\frac{b}{a^2}\,x^2\right]\right)dx$$

$$=\int_0^a \left(\left[\frac{b}{2}\left(\frac{x^2}{a}+x\right)\right]-\left[\frac{b}{a^2}\,x^3\right]\right)dx=\left[\left[\frac{b}{2}\left(\frac{x^3}{3a}+\frac{x^2}{2}\right)\right]-\left[\frac{b}{a^2}\,\frac{x^4}{4}\right]\right]_0^a=\frac{ba^2}{6}$$

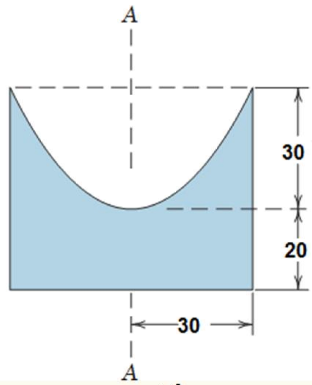
$$\bar{x} = \frac{\int x\,dA}{A} = \frac{\frac{ba^2}{6}}{\frac{5}{12}ba} = \frac{2}{5}a$$

$$M_x = \int y\,dA = \int_0^a \left(\frac{y1+y2}{2}\right)(y2-y1)dx = \frac{1}{2}\int_0^a [{y_2}^2 - {y_1}^2]dx$$

$$= \frac{1}{2}\int_0^a \left[\left(\frac{b}{2}\left(\frac{x}{a}+1\right)\right)^2 - \left(\frac{b}{a^2}x^2\right)^2\right]dx = \frac{23ab^2}{120}$$

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{\frac{23ab^2}{120}}{\frac{5}{12}ba} = \frac{23}{50}b$$

Prob. 5/31



The figure represents a flat piece of sheet metal symmetrical about axis A-A and having a parabolic upper boundary. Choose your own coordinates and calculate the distance from the base to the center of gravity of the piece.

Solution:

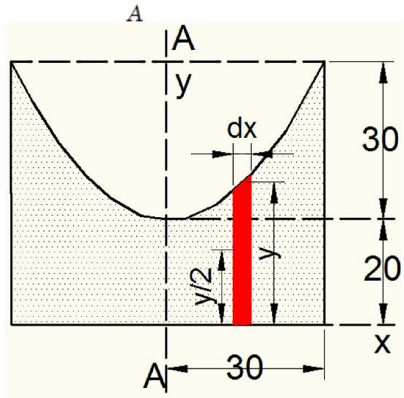
$y = a_0 + a_1 x + a_2 x^2$

$\grave{y} = a_1 + 2a_2 x$

$\grave{y}(0) = 0 = a_1 + 2a_2 * 0 \rightarrow a_1 = 0$

$y(0) = 20 = a_0 + a_2 * 0 \rightarrow a_0 = 20$

$y(30) = 50 = 20 + a_2 * 30^2 \rightarrow a_2 = \frac{1}{30}$



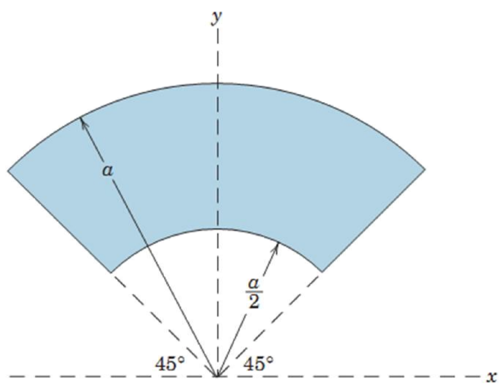
$$y = 20 + \frac{1}{30} * x^2$$

$$A = \int_{-30}^{30} ydx = 2\int_0^{30} ydx = 2\int_0^{30}\left(20+\frac{x^2}{30}\right)dx = 1800\ mm^2$$

$$M_x = \int y\,dA = \int \frac{y}{2}\overbrace{ydx}^{dA} = 2\int_0^{30}\frac{y^2}{2}dx = \int_0^{30}\left(20+\frac{x^2}{30}\right)^2 dx = 29400\ mm^3$$

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{29400}{1800} = 16.33\ mm$$

Prob.5/29



Determine the y-coordinate of the centroid of the shaded area.

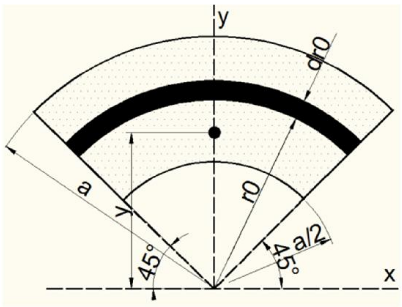
Solution:

$$A = \int_{\frac{a}{2}}^{a} \frac{2\pi r_0}{4}dr_0 = \frac{\pi}{2}\left[\frac{{r_0}^2}{2}\right]_{\frac{a}{2}}^{a} = \frac{3}{16}\pi a^2$$

$$M_x = \int y\,dA = \int \overbrace{\frac{r_0 \sin\alpha}{\alpha}}^{y}\, \overbrace{\frac{2\pi r_0}{4} dr_0}^{dA} = \int_{\frac{a}{2}}^{a} \frac{r_0 \sin\frac{\pi}{4}}{\frac{\pi}{4}} * \frac{2\pi r_0}{4} dr_0$$

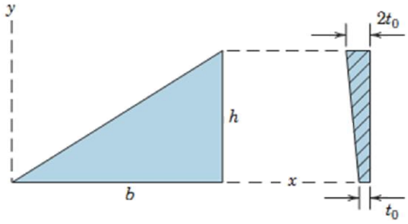
$$= 2\int_{\frac{a}{2}}^{a} \frac{1}{\sqrt{2}} * r_0{}^2 dr_0 = \frac{7}{12\sqrt{2}} a^3$$

$$\bar{y} = \frac{\int y\,dA}{A} = \frac{\frac{7}{12\sqrt{2}}a^3}{\frac{3}{16}\pi a^2} = \frac{289}{9\sqrt{2}}\frac{a}{\pi}$$



Prob. 5/36

The thickness of the triangular plate varies linearly with y from a value t_0 along its base y=0 to 2 t_0 at y=h. Determine the y-coordinate of the center of *mass* of the plate.



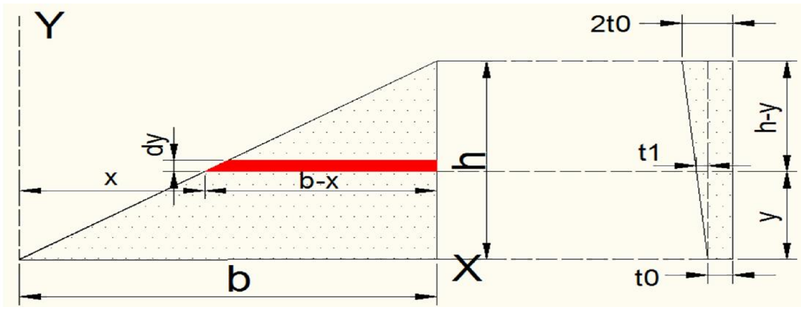
Solution:

$$\frac{b-x}{h-y} = \frac{b}{h} \to x = \frac{b}{h}y$$

$$\frac{t_1}{y} = \frac{t_0}{h} \to t_1 = \frac{t_0}{h}y$$

$$t = t_0 + t_1 = t_0 + \frac{t_0}{h}y$$

$$= t_0(1+\frac{y}{h})$$



$$\bar{y} = \frac{\int y\,dm}{m} = \frac{\int y\,\rho dV}{\int \rho dV} = \frac{\int y\,\rho t dA}{\int \rho t dA}$$

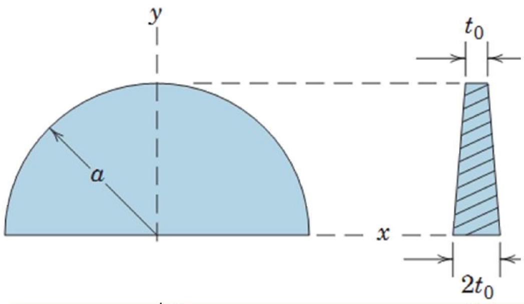
$$\mathrm{m} = \int \rho t dA = \rho \int_0^h \left[t_0\left(1+\frac{y}{h}\right)\right](b-x)dy = t_0\rho \int_0^h \left(1+\frac{y}{h}\right)\left(b-\frac{b}{h}y\right)dy = \frac{2}{3}t_0\rho b h$$

$$M_x = \int y\,\rho t dA = \rho \int_0^h y\left[t_0\left(1+\frac{y}{h}\right)\right](b-x)dy = t_0\rho \int_0^h y\left(1+\frac{y}{h}\right)\left(b-\frac{b}{h}y\right)dy = \frac{t_0\rho b h^2}{4}$$

$$\bar{y} = \frac{\int y\,\rho t dA}{\int \rho t dA} = \frac{\frac{t_0 \rho b h^2}{4}}{\frac{2}{3}t_0\rho b h} = 0.375\,h \text{ Compare with } \bar{y} = \frac{h}{3} \text{ for uniform thickness}$$

Prob.5/42

The thickness of the semicircular plate varies linearly with y from a value $2t_0$ along its base y=0 to t_0 at y=a. Determine the y-coordinate of the *mass* center of the plate.

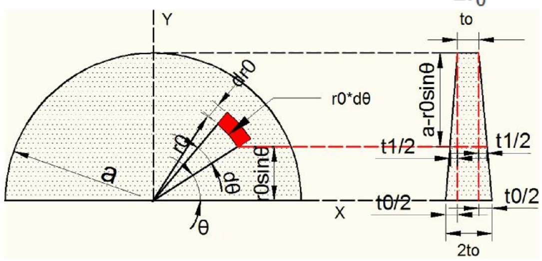


Solution:

$$\frac{t_1/2}{a - r_0\sin\theta} = \frac{t_0/2}{a} \to t_1 = t_0(1-\frac{r_0}{a}\sin\theta)$$

$$t(r,\theta) = t_0 + t_1 = t_0 + \left[t_0(1-\frac{r_0}{a}\sin\theta)\right]$$

$$= t_0(2-\frac{r_0}{a}\sin\theta)$$



$$m = \int \rho t dA = \int_0^a \int_0^{\frac{\pi}{2}} \rho \overbrace{\left[t_0\left(2 - \frac{r_0}{a}\sin\theta\right)\right]}^{t} \overbrace{r_0 d\theta\, dr_0}^{dA} = \rho t_0 \int_0^a \int_0^{\frac{\pi}{2}} \left(2r_0 - \frac{r_0^2}{a}\sin\theta\right) d\theta\, dr_0$$

$$= \rho t_0 \int_0^{\frac{\pi}{2}} \left[\frac{2r_0^2}{2} - \frac{r_0^3}{3a}\sin\theta\right]_0^a d\theta = \rho t_0 a^2 \int_0^{\frac{\pi}{2}} \left(1 - \frac{\sin\theta}{3}\right) d\theta = \rho t_0 a^2 \left[\theta + \frac{\cos\theta}{3}\right]_0^{\frac{\pi}{2}}$$

$$= \rho t_0 a^2 \left(\frac{\pi}{2} - \frac{1}{3}\right) * \underbrace{2}_{two\ quarter-circular\ area} = 2.475\, \rho t_0 a^2$$

$$M_x = \int y\, \rho t dA = \int_0^a \int_0^{\frac{\pi}{2}} \overbrace{r_0 \sin\theta}^{y}\, \rho \overbrace{\left[t_0\left(2 - \frac{r_0}{a}\sin\theta\right)\right]}^{t} \overbrace{r_0 d\theta\, dr_0}^{dA}$$

$$= \rho t_0 \int_0^a \int_0^{\frac{\pi}{2}} \left(2r_0^2 \sin\theta - \frac{r_0^3}{a}\sin^2\theta\right) d\theta\, dr_0$$

$$= \rho t_0 \int_0^{\frac{\pi}{2}} \left[\frac{2r_0^3}{3}\sin\theta - \frac{r_0^4}{4a}\sin^2\theta\right]_0^a d\theta = \rho t_0 \int_0^{\frac{\pi}{2}} \left(\frac{2a^3}{3}\sin\theta - \frac{a^3}{4}\overbrace{\sin^2\theta}^{\frac{1-\cos 2\theta}{2}}\right) d\theta$$

$$= \rho t_0 \int_0^{\frac{\pi}{2}} \left[\frac{2a^3}{3}\sin\theta - \frac{a^3}{4}\left(\frac{1 - \cos 2\theta}{2}\right)\right] d\theta$$

$$= \rho t_0 a^3 \left[-\frac{2}{3}\cos\theta - \frac{1}{8}\left(\theta - \frac{\sin 2\theta}{2}\right)\right]_0^{\frac{\pi}{2}} = \rho t_0 a^3 \left(-\frac{\pi}{16} + \frac{2}{8}\right) * 2$$

$$= 0.9406\, \rho t_0 a^3$$

$$\bar{y} = \frac{\int y\, \rho t dA}{\int \rho t dA} = \frac{0.9406\, \rho t_0 a^3}{2.475\, \rho t_0 a^2} = 0.38\, a \text{ Compare with } \bar{y} = \frac{4r}{3\pi} = 0.424\, a \text{ for uniform thickness}$$

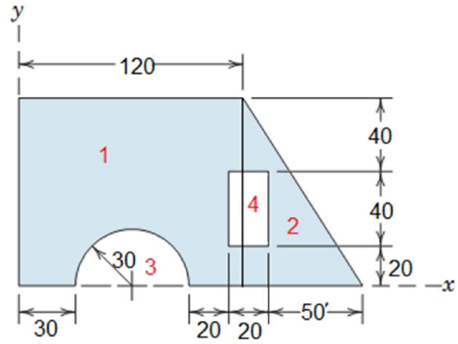
Centroid of Composite Figures

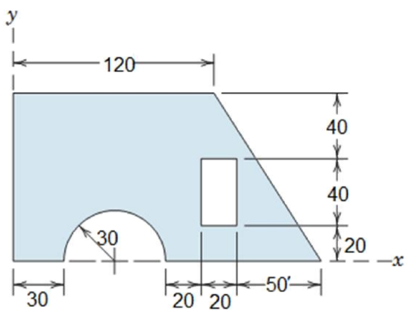
$$\bar{x} = \frac{\sum A * x}{\sum A}$$

$$\bar{y} = \frac{\sum A * y}{\sum A}$$

SAMPLE PROBLEM 5/6

Locate the centroid of the shaded area.





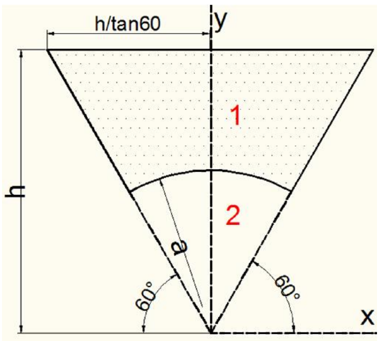
Part	A(mm²)	x(mm)	y(mm)	A*x	A*y
1 rect.	120*100=12000	120/2=60	100/2=50	12000*6=720000	12000*50=600000
2 tri.	(60*100)/2=3000	120+(60/3)=140	100/3=33.3	420000	100000
3 cir	$(30^2*\pi)/2$=-144	30+30=60	$\frac{2}{3}\frac{r\sin\alpha}{\alpha} = \frac{2}{3}\frac{30\sin\frac{\pi}{2}}{\frac{\pi}{2}} = 12.73$	-84800	18000
4 rect	20*40=-800	30+30+30+20+(20/2)=120	20+(40/2)=40	-96000	-32000
Total	12790			959000	650000

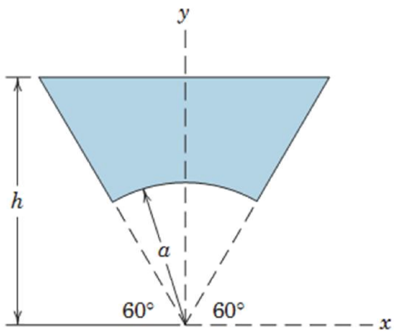
$$\bar{x}=\frac{\sum A*x}{\sum A}=\frac{959000}{12790}=75\ mm$$

$$\bar{y}=\frac{\sum A*y}{\sum A}=\frac{650000}{12790}=50.8\ mm$$

Prob.5/50

Determine the y-coordinate of the centroid of the shaded area.



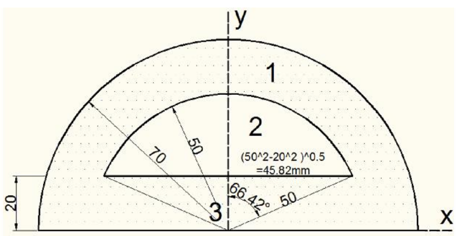


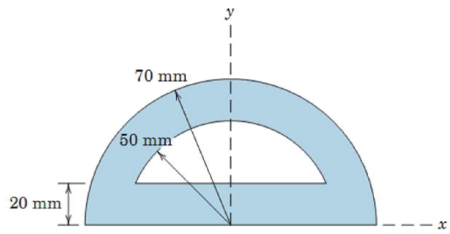
Part	A(mm²)	x(mm)	y(mm)	A*x	A*y
1 tri.	$\frac{h}{\tan 60}*h=\frac{h^2}{\tan 60}$	0	$\frac{2}{3}h$	0	$\frac{2}{3}\frac{h^3}{\tan 60}$
2 *circular sector*	$\frac{a^2\pi}{2\pi}\frac{\pi}{3}=-\frac{a^2\pi}{6}$	0	$\frac{2}{3}\frac{r\sin\alpha}{\alpha}=\frac{2}{3}\frac{a\sin\frac{\pi}{6}}{\frac{\pi}{6}}=\frac{2}{\pi}a$	0	$-\frac{a^3}{3}$
Total					

$$\bar{y}=\frac{\sum A*y}{\sum A}=\frac{\frac{2}{3}\frac{h^3}{\tan 60}-\frac{a^3}{3}}{\frac{h^2}{\tan 60}-\frac{a^2\pi}{6}}=\frac{4h^3-2\sqrt{3}\,a^3}{6h^2-\sqrt{3}\pi a^2}$$

Prob.5/75

Determine the y-coordinate of the centroid of the shaded area.





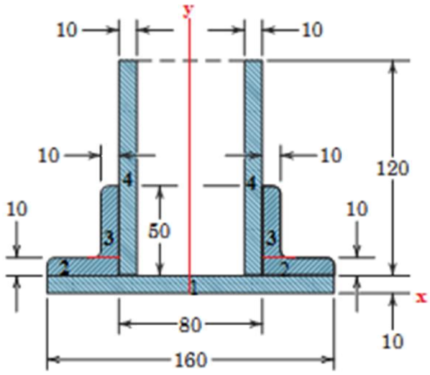
Part	A(mm²)	x(mm)	y(mm)	A*x	A*y
1 *semicircular*	$\frac{70^2\pi}{2}=7697$	0	$\frac{4r}{3\pi}=\frac{4*70}{3\pi}=29.71$	0	228670
2 *circular sector*	$\frac{50^2\pi}{360}*132.84=-2898$	0	$\frac{2}{3}\frac{r\sin\alpha}{\alpha}=\frac{2}{3}\frac{50\sin 66.42}{1.159}=26.35$	0	-76365
3 Triangle	45.82*20=916.4	0	$\frac{2}{3}$*20=13.33	0	12218.7
Total	5715				164523

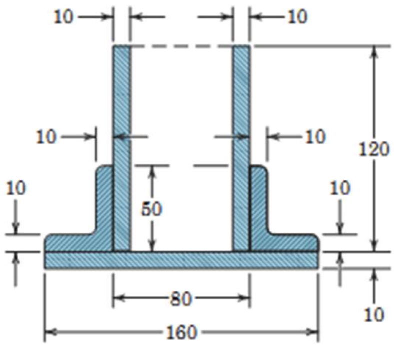
Solution:

$$\theta=\cos^{-1}\frac{20}{50}=66.42°$$

$$\bar{y}=\frac{\sum A*y}{\sum A}=\frac{164523}{5715}=28.8\ \text{mm}$$

Problem 5/57 :Determine the distance from the bottom of the base plate to the centroid of the built-up structural section shown.



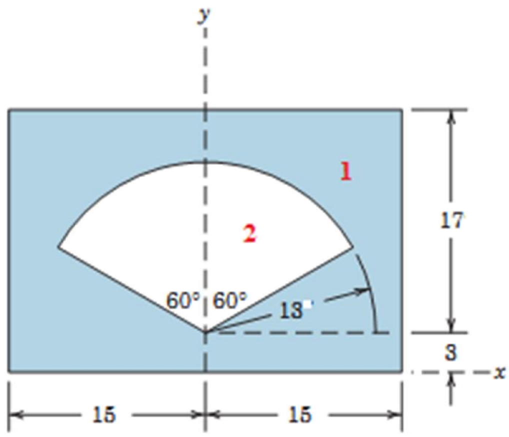


Dimensions in millimeters

Part	A(mm²)	X (mm)	y(mm)	A*x	A*y
1 *Rectangle*	$160 * 10 = 1600$	0	$\frac{10}{2} = 5$	0	8000
2 *Rectangle*	$(160 - 80) * 10 = 800$	0	$10 + \frac{10}{2} = 15$	0	12000
3 *Rectangle*	$((50 - 10) * 10) * 2 = 800$	0	$10 + 10 + \frac{40}{2} = 40$	0	32000
4 *Rectangle*	$(120 * 10) * 2 = 2400$		$10 + \frac{120}{2} = 70$		168000
Total	5600				220000

$$\bar{y} = \frac{\sum A * y}{\sum A} = \frac{220000}{5600} = 39.3mm$$

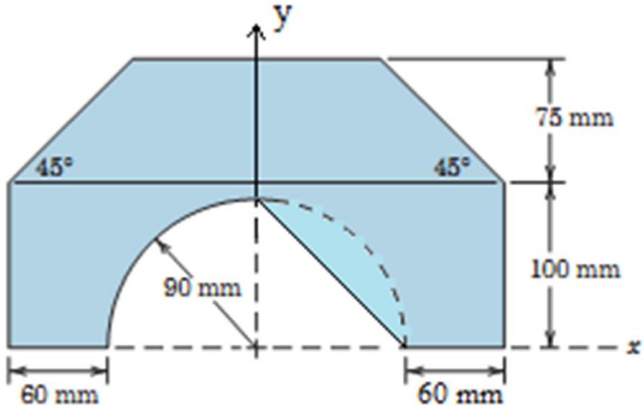
Problem 5/52 :Calculate the y-coordinate of the centroid of the shaded area.



Part	A(mm²)	x(mm)	y(mm)	A*x	A*y
1 *Rectangle*	$20 * 30 = 600$	0	$\frac{20}{2} = 10$	0	6000
2 *circular sector*	$13^2 * \frac{\pi}{3} = -177$	0	$3 + \frac{2}{3}\frac{r \sin\alpha}{\alpha} = 3 + \frac{2}{3} * \frac{13 \sin 60}{\pi/3} = 10.17$	0	-1800
Total	423				4200

$$\bar{y} = \frac{\sum A * y}{\sum A} = \frac{4200}{423} = 9.mm$$

Problem: Locate the centroid of the shaded area shown in Fig.



Solution:

Part	A(mm^2)	x(mm)	y(mm)	A*x	A*y
1 Rec.	150*75=11250	0	100+(75/2)=137.5	0	421875
2 Tri.	(75*75)/2=2812.5	(150/2)+(75/3)=100	100+(75/3)=125	281250	351562.5
3 Tri.	(75*75)/2=2812.5	(150/2)+(75/3)= -100	100+(75/3)=125	-281250	351562.5
4 Rec.	300*100=30000	0	100/2=50	0	1500000
5 Tri,	(90*90)/2=- 4050	90/3=30	90/3=30	-121500	-121500
6 Quarter circular	$(\pi/4)*90^2$=-6361.7	$\frac{4r}{3\pi}=\frac{4*90}{3\pi}$= - 38.2	$\frac{4r}{3\pi}=\frac{4*90}{3\pi}$= 38.2	243017	-243017
Total	36463.3			121517	2260483

$$\bar{x}=\frac{\sum A*x}{\sum A}=\frac{121517}{36463.3}=3.33\ mm$$

$$\bar{y}=\frac{\sum A*y}{\sum A}=\frac{2260483}{36463.3}=62\ mm$$

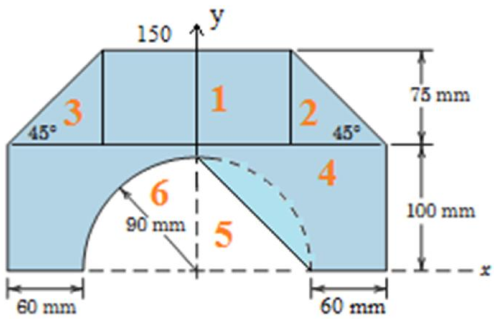


TABLE D/3 PROPERTIES OF PLANE FIGURES

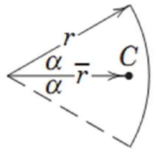
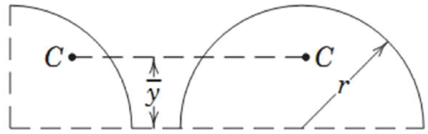
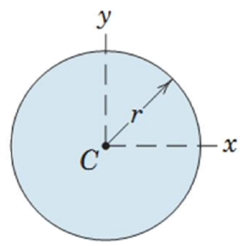
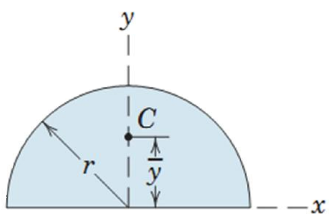
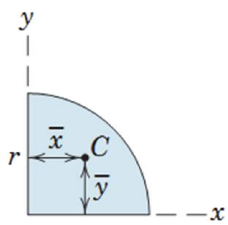
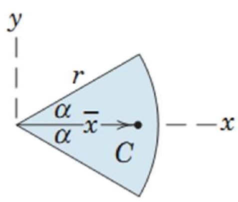
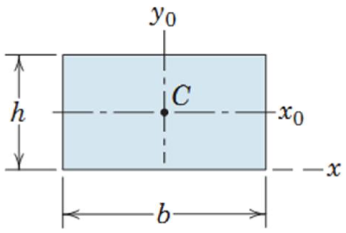
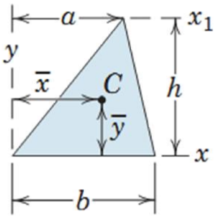
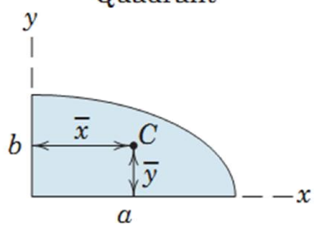
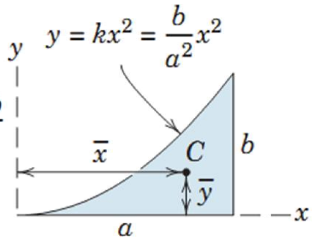
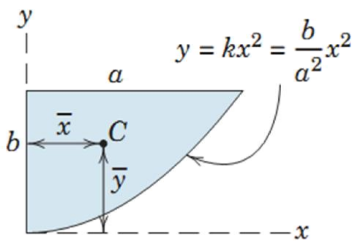
FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Arc Segment 	$\bar{r} = \dfrac{r\sin\alpha}{\alpha}$	—
Quarter and Semicircular Arcs 	$\bar{y} = \dfrac{2r}{\pi}$	—
Circular Area 	—	$I_x = I_y = \dfrac{\pi r^4}{4}$ $I_z = \dfrac{\pi r^4}{2}$
Semicircular Area 	$\bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{8}$ $\bar{I}_x = \left(\dfrac{\pi}{8} - \dfrac{8}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{4}$
Quarter-Circular Area 	$\bar{x} = \bar{y} = \dfrac{4r}{3\pi}$	$I_x = I_y = \dfrac{\pi r^4}{16}$ $\bar{I}_x = \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right) r^4$ $I_z = \dfrac{\pi r^4}{8}$
Area of Circular Sector 	$\bar{x} = \dfrac{2}{3}\dfrac{r\sin\alpha}{\alpha}$	$I_x = \dfrac{r^4}{4}(\alpha - \dfrac{1}{2}\sin 2\alpha)$ $I_y = \dfrac{r^4}{4}(\alpha + \dfrac{1}{2}\sin 2\alpha)$ $I_z = \dfrac{1}{2} r^4 \alpha$

TABLE D/3 PROPERTIES OF PLANE FIGURES *Continued*

FIGURE	CENTROID	AREA MOMENTS OF INERTIA
Rectangular Area 	—	$I_x = \dfrac{bh^3}{3}$ $\bar{I}_x = \dfrac{bh^3}{12}$ $\bar{I}_z = \dfrac{bh}{12}(b^2 + h^2)$
Triangular Area 	$\bar{x} = \dfrac{a+b}{3}$ $\bar{y} = \dfrac{h}{3}$	$I_x = \dfrac{bh^3}{12}$ $\bar{I}_x = \dfrac{bh^3}{36}$ $I_{x_1} = \dfrac{bh^3}{4}$
Area of Elliptical Quadrant 	$\bar{x} = \dfrac{4a}{3\pi}$ $\bar{y} = \dfrac{4b}{3\pi}$	$I_x = \dfrac{\pi ab^3}{16}, \quad \bar{I}_x = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)ab^3$ $I_y = \dfrac{\pi a^3 b}{16}, \quad \bar{I}_y = \left(\dfrac{\pi}{16} - \dfrac{4}{9\pi}\right)a^3 b$ $I_z = \dfrac{\pi ab}{16}(a^2 + b^2)$
Subparabolic Area Area $A = \dfrac{ab}{3}$ 	$\bar{x} = \dfrac{3a}{4}$ $\bar{y} = \dfrac{3b}{10}$	$I_x = \dfrac{ab^3}{21}$ $I_y = \dfrac{a^3 b}{5}$ $I_z = ab\left(\dfrac{a^2}{5} + \dfrac{b^2}{21}\right)$
Parabolic Area Area $A = \dfrac{2ab}{3}$ 	$\bar{x} = \dfrac{3a}{8}$ $\bar{y} = \dfrac{3b}{5}$	$I_x = \dfrac{2ab^3}{7}$ $I_y = \dfrac{2a^3 b}{15}$ $I_z = 2ab\left(\dfrac{a^2}{15} + \dfrac{b^2}{7}\right)$