Applications of Partial Derivatives

1. Jacobian and Hessian Matrices:

The **Jacobian Matrix** is a matrix composed of the first-order partial derivatives of a multivariable function $f(x, y) = \big(u(x, y),\ v(x, y)\big)$. The formula for the Jacobian matrix is the following:

$$\mathrm{J} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix}$$

The determinant of Jacobian matrix is denoted by $|\mathrm{J}| = u_x v_y - v_x u_y$

Example 1: Let $u = x^2 - y^2, v = 2xy$. Find Jacobian matrix and its determinant.

$$u_x = 2x, u_y = -2y, v_x = 2y \text{ and } v_y = 2x$$

$$\mathrm{J} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$|\mathrm{J}| = 2x \times 2x - 2y \times (-2y) = 4x^2 + 4y^2$$

The Jacobian matrix of the function with 3 variables $f(x, y, z) = (u, v, w)$

$$\mathrm{J} = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix}$$

Example 2: Let $u = x^2 y, v = y^2 z,\ \ w = xz^2$. Find Jacobian matrix and its determinant at the point (1,1,1).

$$\mathrm{J} = \begin{bmatrix} 2xy & x^2 & 0 \\ 0 & 2yz & y^2 \\ z^2 & 0 & 2xz \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$$

$$|\mathrm{J}| = \begin{vmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 2 \end{vmatrix} \begin{matrix} 2 & 1 \\ 0 & 2 \\ 1 & 0 \end{matrix}$$

$$= 2 \times 2 \times 2 + 1 \times 1 \times 1 + 0 \times 0 \times 0 - (1 \times 2 \times 0 + 0 \times 1 \times 2 + 2 \times 0 \times 1)$$

$$= 9$$

The second-order partial derivatives of the function $f(x,y)$ can be arranged as a matrix called the **Hessian Matrix**, denoted by $H(f)$:

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The determinant of Hessian matrix is denoted by $|H(f)| = f_{xx}f_{yy} - f_{xy}f_{yx}$

Example 3: Find Hessian matrix and its determinant at the point $(1,0)$.

for the function $f(x,y) = (x^2+y^2)^2/2$

$f_x = 2x(x^2+y^2) = 2x^3 + 2xy^2$

$f_{xx} = 6x^2 + 2y^2$

$f_{xx}(1,0) = 6$

$f_y = 2y(x^2+y^2) = 2yx^2 + 2y^3$

$f_{yy} = 2x^2 + 6y^2$

$f_{yy}(1,0) = 2$

$f_{xy} = f_{yx} = 2y \times 2x = 4xy$

$f_{xy}(1,0) = 0$

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 0 & 2 \end{bmatrix}$$

$$|H(f)| = \begin{vmatrix} 6 & 0 \\ 0 & 2 \end{vmatrix} = 12$$

The Hessian matrix of the function with 3 variables $f(x,y,z)$ is the matrix:

$$H(f) = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

Example 4: Let $f(x,y,z) = x^2e^y + xz^2$. Find Hessian matrix and its determinant at the point $(1,0,2)$.

$f_x = 2xe^y + z^2 \quad ⇨ \quad f_{xx} = 2e^y \quad ⇨ \quad f_{xx}(1,0,2) = 2$

$f_x = 2xe^y + z^2 \quad ⇨ \quad f_{xy} = 2xe^y \quad ⇨ \quad f_{xy}(1,0,2) = 2$

$f_x = 2xe^y + z^2 \quad ⇨ \quad f_{xz} = 2z \quad ⇨ \quad f_{xz}(1,0,2) = 4$

$f_y = x^2e^y \Rightarrow f_{yx} = 2xe^y \Rightarrow f_{yx}(1,0,2) = 2$

$f_y = x^2e^y \Rightarrow f_{yy} = x^2e^y \Rightarrow f_{yy}(1,0,2) = 1$

$f_y = x^2e^y \Rightarrow f_{yz} = 0 \Rightarrow f_{yz}(1,0,2) = 0$

$f_z = 2xz \Rightarrow f_{zx} = 2z \Rightarrow f_{zx}(1,0,2) = 4$

$f_z = 2xz \Rightarrow f_{zy} = 0 \Rightarrow f_{zy}(1,0,2) = 0$

$f_z = 2xz \Rightarrow f_{zz} = 2x \Rightarrow f_{zz}(1,0,2) = 2$

$$H(f) = \begin{bmatrix} 2 & 2 & 4 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

$$|H(f)| = \begin{vmatrix} 2 & 2 & 4 \\ 2 & 1 & 0 \\ 4 & 0 & 2 \end{vmatrix} \begin{matrix} 2 & 2 \\ 2 & 1 \\ 4 & 0 \end{matrix}$$

$$= 2 \times 1 \times 2 + 2 \times 0 \times 4 + 4 \times 2 \times 0 - (4 \times 1 \times 4 + 0 \times 0 \times 2 + 2 \times 2 \times 2)$$

$$= 4 - (16 - 8) = -4$$

2. Gradient and Laplace Operator of a Scalar Field

A scalar field is a function that takes a point in space and assign a number to it, for example $f(x,y,z) = x^2 + \cos 2y + \ln(2z+1)$

$f(1,\pi/6,0) = 1 + \cos(\pi/3) + \ln 1 = 1 + (1/2) + 0 = 3/2$

The Gradient of a scalar field $f(x,y,z)$ is a vector field denoted by ∇f and it is defined as: $\boxed{\nabla f = f_x\, i + f_y\, j + f_z\, k}$

Example 5: Find ∇f of $f(x,y,z) = 2x^2 \sin y - xy \tan z$

$$f_x = 4x \sin y - y \tan z,\quad f_y = 2x^2 \cos y - x \tan z \text{ and } f_z = -xy \sec^2 z$$

$$\nabla f = (4x \sin y - y \tan z)\, i + (2x^2 \cos y - x \tan z)\, j - xy \sec^2 z\ k$$

Laplace Operator: The differential operator ∇^2 is called Laplace operator and it is defined as: $\boxed{\nabla^2 f = f_{xx} + f_{yy} + f_{zz}}$

Example 6: Find ∇f and $\nabla^2 f$ for $f(x,y,z) = x^3e^y + xy^2z^3$

$$f_x = 3x^2e^y + y^2z^3 \Rightarrow f_{xx} = 6xe^y$$

$$f_y = x^3e^y + 2xyz^3 \quad \Rightarrow \quad f_{yy} = x^3e^y + 2xz^3$$

$$f_z = 3xy^2z^2 \quad \Rightarrow \quad f_{zz} = 6xy^2z$$

$$\nabla f = f_x\, i + f_y\, j + f_z\, k = (3x^2e^y + y^2z^3)\, i + (x^3e^y + 2xyz^3)\, j + 3xy^2z^2\, k$$

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz} = 6xe^y + x^3e^y + 2xz^3 + 6xy^2z$$

3. Divergence and the Curl of a Vector Field

The Divergence of a vector field $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ is computed as:

$$div\ F = \nabla . F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example 7: Find $div\ F$ if $F(x,y,z) = xzi + e^{yz}j - \ln(xy)\,k$

$$div\ F = \nabla . F = \frac{\partial(xz)}{\partial x} + \frac{\partial(e^{yz})}{\partial y} - \frac{\partial(\ln(xy))}{\partial z} = z + ze^{yz}$$

The Curl of a vector field $F(x,y,z) = F_1(x,y,z)i + F_2(x,y,z)j + F_3(x,y,z)k$ it is another vector defined as the following determinant:

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) i - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) j + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) k$$

Example 8: Find $curl\ F$ if $F(x,y,z) = xyi + yzj + xzk$ at $(-1,-3,-2)$

$$curl\ F = \nabla \times F = \begin{vmatrix} i & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$

$$= \left(\frac{\partial(xz)}{\partial y} - \frac{\partial(yz)}{\partial z}\right) i - \left(\frac{\partial(xz)}{\partial x} - \frac{\partial(xy)}{\partial z}\right) j + \left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right) k$$

$$= -yi - zj - xk$$

$$curl\ F\ \Big|_{\text{at }(-1,-3,-2)} = 3i + 2j + k$$

Exercises

1. Let $u = x\cos y$, $v = x\sin y$. Find Jacobian matrix and its determinant at $(1, \pi/4)$.
2. Let $u = xe^y, v = ye^z, \ \ w = ze^x$. Find Jacobian matrix and its determinant.
3. Find the determinant of the Hessian matrix for the functions

   $(a) \quad f(x, y) = x^2y + xy^2.$

   $(b) \quad f(x, y, z) = x^2 \sin(yz)$
4. If $f(x, y, z) = x^3y^2z$, then find

   $(a) \quad \nabla f \quad \text{at} \quad (-1,2,-2)$

   $(b) \quad \nabla^2 f \ \text{at} \quad (1,-3,2)$
5. If $F(x, y, z) = yze^{xy}i + xze^{xy}j + (e^{xy} + 3\cos 3z)k$, then find

   $(a) \quad div\, F \ \text{at} \quad \left(0, \sqrt{6}, \pi/6\right)$

   $(b) \quad curl\, F$