Continuous Probability Distributions (التوزيعات الاحتمالية المستمرة)

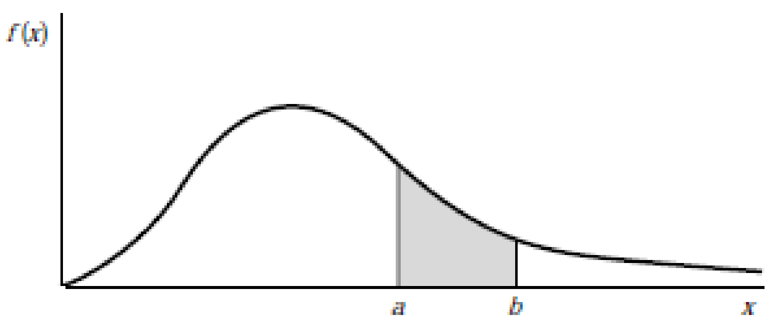
Let's now consider the distributions of continuous random variables. A continuous variable is one that can take any value within a specified range of values. Therefore, between any two values a continuous variable takes, there are an infinite number of values.

A nonnegative function $f\,(x)$ is called a probability distribution (sometimes called a probability density function) of the continuous random variable X if the total area bounded by its curve and the x $-$axis is equal to 1

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

The perpendiculars erected at any two points a and b give the probability that X is between the points a and b.

$$P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx$$

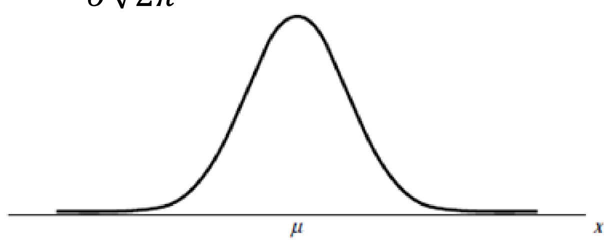


Graph of a continuous distribution showing area between a and b.

Normal Distribution (التوزيع الطبيعي)

The most important distribution in statistics is the Normal distribution, often called the Gaussian distribution. The probability density function of the continuous random variable X with mean μ and variance σ^2 symbolize as $X \sim N(\mu, \sigma^2)$, it is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \cdots (1)$$



Graph of a normal distribution.

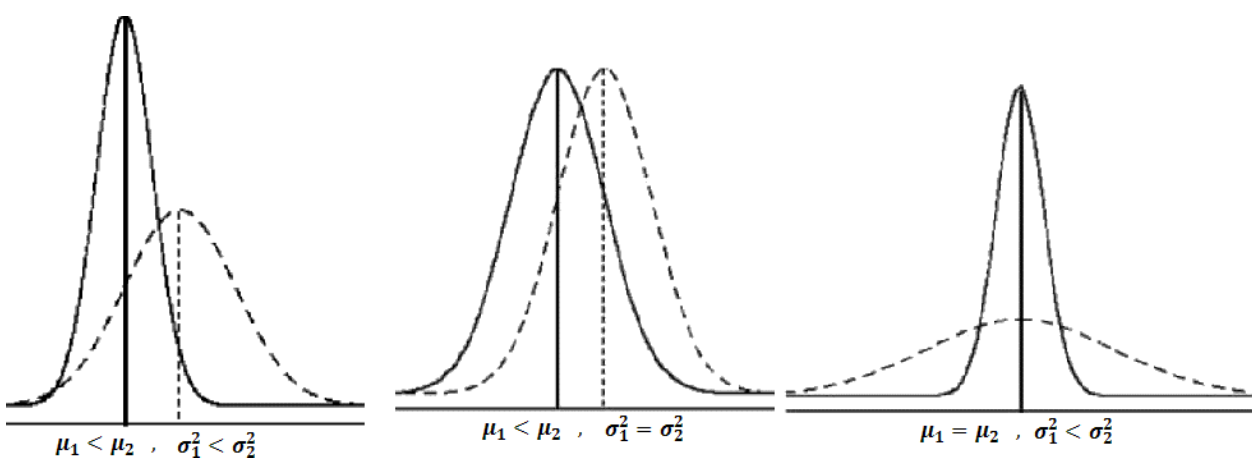
Properties of the Normal Distribution

1. It is symmetrical about its mean μ.
2. The mean, the median, and the mode are all equal.
3. The total area under the curve above the x −axis is one square unit.

$$\int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx = 1$$

4. The probability density function depends on the mean and variance.

The figure below shows the probability density function for two random variables $X_1 \sim N(\mu_1, \sigma_1^2)$ (drawn with solid lines) and $X_2 \sim N(\mu_2, \sigma_2^2)$ (drawn with dashed lines).



Standard Normal Distribution (التوزيع الطبيعي المعياري)

The standard normal distribution, or unit normal distribution, as it is sometimes called, has a mean of 0 and a variance of 1. i.e $Z \sim N\ (0\ ,1)$.

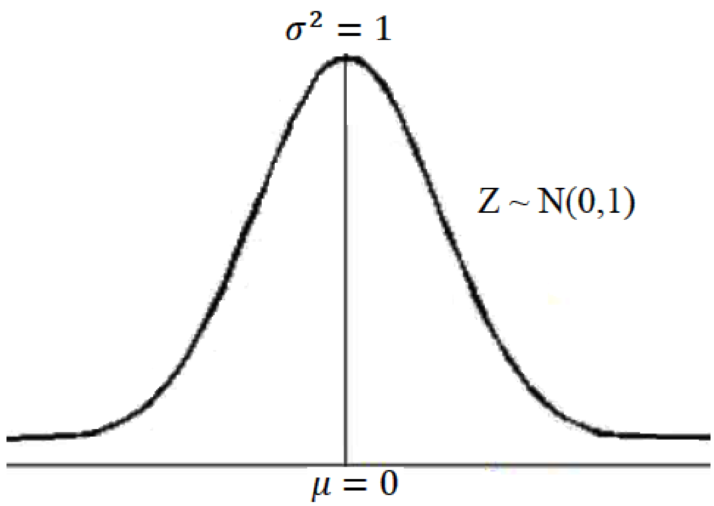
It may be obtained from Equation (1) by creating a random variable.

$$z = \frac{x - \mu}{\sigma} \qquad \cdots (2)$$

The equation for the standard normal distribution is written:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \qquad \cdots (3)$$

The figure below shows the probability density function for the random variable Z.



There is a special table for the standard normal distribution, through which the probability of a random variable with a normal distribution can be found. The following are examples to illustrate how to find the required probability:

1. How to find $P(z \leq z_1)$. We illustrate this with the example $P(z \leq -1.94\ ) = 0.0262$.

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.5	0.0062	0.006	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.008	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.011
-2.1	0.0179	0.0174	0.017	0.0166	0.0162	0.0158	0.0154	0.015	0.0146	0.0143
-2	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.025	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294

2. $P(z > z_1) = 1 - P(z \leq z_1)$. For example, $P(z > -1.94) = 1 - P(z \leq -1.94)$

$$= 1 - 0.0262 = 0.9738$$

3. $P(z_1 \leq z \leq z_2) = P(z \leq z_2) - P(z \leq z_1)$. For example,

$$P(0.82 \leq z \leq 1.26) = P(z \leq 1.26) - P(z \leq 0.82) = 0.8962 - 0.7939 = 0.1023$$

0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015

The normally distributed random variables $X \sim N(\mu, \sigma^2)$ can be transformed into standard normally distributed random variables $Z \sim N(0, 1)$ using the equation (2).

$$z = \frac{x - \mu}{\sigma}$$

Example 1: If $X \sim N(50, 100)$. Then find $P(45 \leq x \leq 62)$.

Solution:

$$z_1 = \frac{x_1 - \mu}{\sigma} = \frac{45 - 50}{\sqrt{100}} = -0.5$$

$$z_2 = \frac{x_2 - \mu}{\sigma} = \frac{62 - 50}{\sqrt{100}} = 1.2$$

$$P(45 \leq x \leq 62) = P(-0.5 \leq z \leq 1.2)$$

$$= P(z \leq 1.2) - P(z \leq -0.5)$$

$$= 0.8849 - 0.3085 = 0.5764$$

Example 2: If the hemoglobin level in the blood of a community is distributed according to the normal distribution with a mean of 16 and a variance of 0.81, then if we choose a person at random, what is the probability that his hemoglobin level will be greater than 17?

Solution: $\sigma = \sqrt{0.81} = 0.9$

$$z = \frac{x - \mu}{\sigma} = \frac{17 - 16}{0.9} = 1.11$$

$$P(x > 17) = P(z > 1.11)$$

$$= 1 - P(z \leq 1.11) = 1 - 0.8665 = 0.1335$$

Example 3: Suppose the number of ridges in the fingerprints of individuals in a given population is normally distributed with mean 140 and standard deviation 50. Find the probability that the number of ridges on the fingers of an individual randomly selected from this population is:

a) 200 or more
b) Less than 100
c) Between 100 and 200
d) In a population of 10,000 people how many would you expect to have a ridge count of more than150?

Solution:

a) $$z = \frac{x-\mu}{\sigma} = \frac{200-140}{50} = 1.2$$

$$P(x \geq 200\ ) = P(z \geq 1.2)$$

$$= 1 - P(z < 1.2) = 1 - 0.8849 = 0.1151$$

b) $$z = \frac{x-\mu}{\sigma} = \frac{100-140}{50} = -0.8$$

$$P(x < 100\ ) = P(z < -0.8) = 0.2119$$

c) $$P(100 \leq x \leq 200\ ) = P(-0.8 \leq z \leq 1.2\ )$$

$$= P(z \leq 1.2) - P(z \leq -0.8)$$

$$= 0.8849 - 0.2119 = 0.673$$

d) $$z = \frac{x-\mu}{\sigma} = \frac{150-140}{50} = 0.2$$

$$P(x > 150\ ) = P(z > 0.2) = 1 - P(z < 0.2)$$

$$= 1 - 0.4207 = 0.5793$$

$$P(x > 150\ ) = \frac{n(x > 150\ )}{n(S)}$$

$$0.5793 = \frac{n(x > 150\,)}{10000}$$

$$n(x > 150\,) = 5793$$

H.W

1. If $X \sim N\,(75\,,625)$. Then find:

(a) $P(50 \leq x \leq 100\,)$

(b) $P(x > 60\,)$

(c) $P(x < 90\,)$

2. If the total cholesterol values for a certain population are approximately normally distributed with a mean of 200 mg /100 ml and a standard deviation of 20 mg/100 ml, find the probability that an individual picked at random from this population will have a cholesterol value:

(a) Between 180 and 200 mg/100 ml

(b) Greater than 225 mg/100 ml

(c) Less than 150 mg/100 ml

(d) Between 190 and 210 mg/100 ml

Z	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
-2.5	0.0062	0.006	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
-2.4	0.0082	0.008	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
-2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
-2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.011
-2.1	0.0179	0.0174	0.017	0.0166	0.0162	0.0158	0.0154	0.015	0.0146	0.0143
-2	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
-1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.025	0.0244	0.0239	0.0233
-1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
-1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
-1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
-1.5	0.0668	0.0655	0.0643	0.063	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
-1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
-1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
-1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.102	0.1003	0.0985
-1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.123	0.121	0.119	0.117
-1	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
-0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.166	0.1635	0.1611
-0.8	0.2119	0.209	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
-0.7	0.242	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
-0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
-0.5	0.3085	0.305	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.281	0.2776
-0.4	0.3446	0.3409	0.3372	0.3336	0.33	0.3264	0.3228	0.3192	0.3156	0.3121
-0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.352	0.3483
-0.2	0.4207	0.4168	0.4129	0.409	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
-0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
- 0	0.5	0.496	0.492	0.488	0.484	0.4801	0.4761	0.4721	0.4681	0.4641
0	0.5	0.504	0.508	0.512	0.516	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.591	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.648	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.67	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.695	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.719	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.758	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.791	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.834	0.8365	0.8389
1	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.877	0.879	0.881	0.883
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.898	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.937	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.975	0.9756	0.9761	0.9767
2	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.983	0.9834	0.9838	0.9842	0.9846	0.985	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.989
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.992	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.994	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952