3- Integration by using the formula

$$\int u\,dv = u\,v - \int v\,du$$

Which is used for inverse trigonometric functions and logarithm functions.

Example 1: Find $\int \ln x\,dx$

$$u = \ln x \qquad \text{and} \qquad dv = dx$$

$$du = \frac{dx}{x} \qquad \text{and} \qquad v = x$$

$$\int u\,dv = u\,v - \int v\,du$$

$$\int \ln x\,dx = x\ln x - \int x \times \frac{dx}{x}$$

$$= x\ln x - \int dx = x\ln x - x + c$$

Example 2: Find $\int \sin^{-1} 2x\,dx$

$$u = \sin^{-1} 2x \qquad \text{and} \qquad dv = dx$$

$$du = \frac{2dx}{\sqrt{1-4x^2}} \qquad \text{and} \qquad v = x$$

$$\int \sin^{-1} 2x\,dx = x\sin^{-1} 2x - \int \frac{2xdx}{\sqrt{1-4x^2}}$$

$$= x\sin^{-1} 2x + \frac{1}{2}\sqrt{1-4x^2} + c$$

4- Integration by Partial Fractions

Cover-up Method: We will use the Cover-up method to a rapid method for performing partial fraction whose denominator is a polynomial.

In middle school, you studied adding or subtracting fractions. Let us remember this together with the following example:

$$\frac{1}{(x-3)}+\frac{2}{(x+2)}=\frac{(x+2)+2(x-3)}{(x-3)(x+2)}$$

$$=\frac{3x-4}{x^2-x-6}$$

But how do we go in the opposite direction?

$$\frac{3x-4}{x^2-x-6}\overset{??}{=}\frac{1}{(x-3)}+\frac{2}{(x+2)}$$

Step 1: Analyze the denominator of the fraction:

$$\frac{3x-4}{x^2-x-6}=\frac{3x-4}{(x-3)(x+2)}$$

Step 2: Split the fraction into two parts as follows:

$$\frac{3x-4}{(x-3)(x+2)}=\frac{A}{(x-3)}+\frac{B}{(x+2)}$$

Step 3: Find the values of A and B as following:

To find A : put $x=3$ and cover up the factor $(x-3)$

$$A=\frac{3x-4}{(x-3)(x+2)}=\frac{3\times 3-4}{(3+2)}=1$$

To find B : put $x=-2$ and cover up the factor $(x+2)$

$$B=\frac{3x-4}{(x-3)(x+2)}=\frac{3\times(-2)-4}{(-2-3)}=2$$

Thus, our answer is:

$$\frac{3x-4}{x^2-x-6}=\frac{1}{(x-3)}+\frac{2}{(x+2)}$$

The rational functions we will consider here for integration purposes are those whose denominators can be factored into only two linear factors.

Example 3: Evaluate $\int \frac{3x-4}{x^2-x-6}dx$

$$\int \frac{3x-4}{x^2-x-6}dx = \int \left(\frac{1}{(x-3)} + \frac{2}{(x+2)}\right)dx$$

$$= \ln|x-3| + 2\ln|x+2| + c$$

Example 4: Evaluate $\int_2^3 \frac{dx}{1-x^2}$

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{(1-x)} + \frac{B}{(1+x)}$$

To find A : put $x = 1$ and cover up the factor $(1-x)$

$$A = \frac{1}{(1-x)(1+x)} = \frac{1}{1+1} = \frac{1}{2}$$

To find B : put $x = -1$ and cover up the factor $(1+x)$

$$B = \frac{1}{(1-x)(1+x)} = \frac{1}{1-(-1)} = \frac{1}{2}$$

Thus, our answer is:

$$\frac{1}{1-x^2} = \frac{1/2}{(1-x)} + \frac{1/2}{(1+x)}$$

Then,

$$\int_2^3 \frac{dx}{1-x^2} = \int_2^3 \left(\frac{1/2}{(1-x)} + \frac{1}{(1+x)}\right)dx$$

$$= -\frac{1}{2}\ln|1-x| + \frac{1}{2}\ln|1+x|\Big|_2^3$$

$$= \frac{1}{2}\ln\frac{|1+x|}{|1-x|}\Big|_2^3 = \frac{1}{2}\left(\ln\frac{4}{2} - \ln\frac{3}{1}\right) = \frac{1}{2}(\ln 2 - \ln 3)$$

Exercises

1. $\int x \ln x \, dx$ 2. $\int \tan^{-1} 2x \, dx$ 3. $\int \frac{3x+5}{x^2-2x-15}dx$