Methods of integration

1- Substitution Rule

Most integration problems we will encounter will not be so simple. That is to say we will require more than the basic integration rules we have seen.

Here's a slightly more complicated example:

Example1: Find $\int \frac{xdx}{\sqrt{x^2-3}}$

Let $u = x^2 - 3 \;\Rightarrow\; du = 2xdx$

$$\int \frac{xdx}{\sqrt{x^2-3}} = \int \frac{du}{2\sqrt{u}}$$

$$= \frac{1}{2}\int u^{\frac{-1}{2}}\,du = u^{\frac{1}{2}} + C$$

$$\int \frac{xdx}{\sqrt{x^2-3}} = \sqrt{x^2-3} + C$$

Example2: Find $\int \frac{2xdx}{\sqrt[3]{x^2+1}}$

Let $u = x^2 + 1 \;\Rightarrow\; du = 2xdx$

$$\int \frac{2xdx}{\sqrt[3]{x^2+1}} = \int \frac{du}{\sqrt[3]{u}}$$

$$= \int u^{-1/3}du = \frac{3}{2}u^{2/3} + C$$

$$\int \frac{2xdx}{\sqrt[3]{x^2+1}} = \frac{3}{2}(x^2+1)^{2/3} + C$$

Example3: Evaluate $\int_{1/4}^{1/2} \frac{\cos(\pi x)}{\sin^2(\pi x)}dx$

Let $u = \sin(\pi x) \;\Rightarrow\; du = \pi\cos(\pi x)\,dx \;\Rightarrow\; \cos(\pi x)\,dx = \frac{du}{\pi}$

When $x = \frac{1}{4} \;\Rightarrow\; u = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $x = \frac{1}{2} \;\Rightarrow\; u = \sin\frac{\pi}{2} = 1$

$$\int_{1/4}^{1/2} \frac{\cos(\pi x)}{\sin^2(\pi x)} dx = \int_{1/\sqrt{2}}^{1} \frac{du}{\pi u^2}$$

$$= \frac{1}{\pi} \int_{1/\sqrt{2}}^{1} u^{-2} \; du$$

$$= \frac{-1}{\pi} u^{-1} \Big|_{1/\sqrt{2}}^{1} = \frac{-1}{\pi u}\Big|_{1/\sqrt{2}}^{1} = \frac{-1}{\pi}(1 - \sqrt{2})$$

2- Tabular Integration Method

Take a look at where we can apply the tabular integration method.

(1) When Integrand is the product of Polynomial times and something that can be repeatedly integrated, for example $\int x^5 \sin 2x \;\; dx$ or $\int x^4 \cos 3x \;\; dx$.

(2) Integrand multiple of power function and an exponential function, for example $\int x^2 e^{5x} \; dx$.

The following steps to integrate using the table in cases (1) and (2):

Step 1: We create a table with 4 columns width, in the first column, switch the signs from (+) to (−) leaving the first row unsigned.

Step 2: In the second column, we leave the first row and put the polynomial function in the second row, its first derivative in the third row, the second derivative in the fourth row, and so on until we reach 0.

Step 3: In the third column, we put the other function in the first row, integrate it in the second row, integrate it again in the third row, and continue until we equal the number of rows in the first column.

Step 4: In the fourth column, we put the product of the contents of the previous three columns.

Step 5: To calculate the integral, we add the contents of the fourth column.

Example4: Find $\int x^2 \cos x \, dx$

$+/-$	D	I	Product
		$\cos x$	
$+$	x^2	$\sin x$	$x^2 \sin x$
$-$	$2x$	$-\cos x$	$2x \cos x$
$+$	2	$-\sin x$	$-2 \sin x$
$-$	0	$\cos x$	0

$$\int x^2 \cos x \, dx = x^2 \sin x + 2x \cos x - 2 \sin x + c$$

$$= (x^2 - 2) \sin x + 2x \cos x + c$$

Example 5: Find $\int x^3 e^{3x} dx$

$+/-$	D	I	Product
		e^{3x}	
$+$	x^3	$1/3\, e^{3x}$	$(1/3)x^3 e^{3x}$
$-$	$3x^2$	$1/9\, e^{3x}$	$-(1/3)x^2 e^{3x}$
$+$	$6x$	$1/27\, e^{3x}$	$(2/9)x e^{3x}$
$-$	6	$1/81\, e^{3x}$	$-(2/27)e^{3x}$
$+$	0	$1/243\, e^{3x}$	0

$$\int x^2 e^{3x} dx = (1/3)x^3 e^{3x} - (1/3)x^2 e^{3x} + (2/9)x e^{3x} - (2/27)e^{3x} + c$$

$$= \frac{e^{3x}}{27}(9x^3 - 9x^2 + 6x - 2) + c$$

Exercises

1. $\int \frac{dx}{\sqrt[3]{1-3x}}$

2. $\int_0^{\pi/3} \frac{\sin x}{\cos^2 x} dx$

3. $\int x^2 e^{-3x} \, dx$