Partial Differential Equation PDE

A differential equation involving partial derivatives of a dependent variable (one or more) with more than one independent variable is called a partial differential equation, hereafter denoted as PDE. For example, the following equations are PDEs:

$$a\frac{\partial u}{\partial t}+b\frac{\partial u}{\partial x}=f(x,t) \quad \equiv \quad au_t+bu_x=f(x,t)$$

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0 \quad \equiv \quad u_{tt}-c^2u_{xx}=0 \qquad \text{(Wave Equation)}$$

$$\frac{\partial u}{\partial t}-T\frac{\partial^2 u}{\partial x^2}=0 \quad \equiv \quad u_t-Tu_{xx}=0 \qquad \text{(Heat Equation)}$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \equiv \quad u_{xx}+u_{yy}=0 \qquad \text{(Laplace Equation)}$$

1. First Order PDE

Let's consider the PDE $a(x,t)u_t+b(x,t)u_x=f(x,t)$ with an initial condition $u(x,0)=g(x)$ where $a(x,t)$, $b(x,t)$, $f(x,t)$ and $g(x)$ are given functions.

The system of ordinary differential equations is:

$$\frac{dt}{a(x,t)}=\frac{dx}{b(x,t)}=\frac{du}{f(x,t)}$$

The following examples explain how to find the solution $u(x,t)$.

Example 1: Solve the initial value problem (IVP) $u_t+tu_x=0$ with $u(x,0)=\cos 2x$

Solution: $\frac{dt}{1}=\frac{dx}{t} \Rightarrow tdt=dx \Rightarrow \int tdt=\int dx$

$$\frac{t^2}{2}+C=x \Rightarrow C=x-\frac{t^2}{2}$$

Therefore $u(x,t)=g\left(x-\frac{t^2}{2}\right)$ is the general solution

$$u(x,0)=\cos 2x \Rightarrow u(x,0)=g(x)=\cos 2x$$

$$u(x,t)=\cos 2\left(x-\frac{t^2}{2}\right)=\cos(2x-t^2)$$

Example 2: Solve the IVP $tu_t + xu_x = 0$ with $u(x,2) = \sin 4x$

Solution: $\frac{dt}{t} = \frac{dx}{x} \Rightarrow \int \frac{dt}{t} = \int \frac{dx}{x}$

$\ln t = \ln x + C_1 \Rightarrow C_1 = \ln x - \ln t$

$C_1 = \ln\left(\frac{x}{t}\right) \Rightarrow e^{C_1} = \frac{x}{t}$

$C = \frac{x}{t}$, where $e^{C_1} = C$

Therefore $u(x,t) = g\left(\frac{x}{t}\right)$ is the general solution

$u(x,2) = \sin 4x \Rightarrow g\left(\frac{x}{2}\right) = \sin 4\left(\frac{x}{2}\right) = \sin 2x$

So, $u(x,t) = \sin(2x/t)$

Example 3: Solve the IVP $xu_t - 2xtu_x = 2tu$ with $u(x,0) = x^3$

Solution: $\frac{dt}{x} = \frac{dx}{-2xt} = \frac{du}{2tu}$

$\frac{dt}{x} = \frac{dx}{-2xt} \Rightarrow 2tdt =- dx$

$t^2 + x = C$

And $\frac{dx}{-2xt} = \frac{du}{2tu} \Rightarrow \frac{dx}{-x} = \frac{du}{u}$

$-\ln x + \ln K = \ln u \Rightarrow u = \frac{K}{x}$

Then $u(x,t) = \frac{g(t^2 + x)}{x}$

$u(x,0) = x^3 \Rightarrow \frac{g(x)}{x} = x^3 \Rightarrow g(x) = x^4$

$u(x,t) = \frac{(t^2 + x)^4}{x}$

Example 4: Solve the IVP $u_t + 4u_x = u^2$ with $u(x,0) = \dfrac{1}{1+x^2}$

Solution: $\dfrac{dt}{1} = \dfrac{dx}{4} = \dfrac{du}{u^2}$

$$\frac{dt}{1} = \frac{dx}{4} \Rightarrow dx = 4dt$$

$$x = 4t + C \Rightarrow C = x - 4t$$

And $\dfrac{dt}{1} = \dfrac{du}{u^2} \Rightarrow t + K = -\dfrac{1}{u}$

$$u = \frac{-1}{t+K}$$

Then $u(x,t) = \dfrac{-1}{t + g(x-4t)}$

$$u(x,0) = \frac{1}{1+x^2} \Rightarrow \frac{-1}{g(x)} = \frac{1}{1+x^2} \Rightarrow g(x) = -(x^2+1)$$

$$u(x,t) = \frac{-1}{t - [(x-4t)^2 + 1]}$$

Example 5: Solve the IVP $u_x + 2u_t = x - u$ with $u(0,t) = e^{2t} - 1$

Solution: $\dfrac{dx}{1} = \dfrac{dt}{2} = \dfrac{du}{x-u}$

$$\frac{dx}{1} = \frac{dt}{2} \Rightarrow C = 2x - t$$

$$\frac{dx}{1} = \frac{du}{x-u} \Rightarrow \frac{du}{dx} = x - u \Rightarrow \frac{du}{dx} + u = x$$

This is a linear first order differential equation. It can be solved using the integrating factor:

$$\mu(x) = e^{\int P(x)dx} = e^{\int dx} = e^x$$

$$u\,\mu(x) = \int \mu(x)\, Q(x)dx \Rightarrow u\, e^x = \int xe^x\, dx = xe^x - e^x + g(C)$$

Then $u(x,t) = x - 1 + e^{-x}\, g(2x-t)$

$$u(0,t) = e^{2t} - 1 \Rightarrow e^{2t} - 1 = -1 + g(-t) \Rightarrow g(-t) = e^{2t} \Rightarrow g(t) = e^{-2t}$$

$$u(x,t) = x - 1 + e^{-x}\, e^{-2(2x-t)} = x - 1 + e^{-5x+2t}$$

H.W: Solve the initial value problems

1. $2u_t + 5u_x = 0$ with $u(x,0) = 1/(1+e^x)$. $Ans: u(x,t) = 1\Big/\left(1+e^{x-\frac{5t}{2}}\right)$

2. $3u_t + 2u_x = \cos t$ with $u(x,0) = \sin x$. $Ans: u(x,t) = \frac{1}{3}\sin t + \sin\left(x - \frac{2}{3}t\right)$

3. $12u_y - 5u_x = 0$ with $u(0,y) = y^2$. $Ans: u(x,y) = \frac{1}{144}(5x+12y)^2$

4. $u_t + u_x = u$ with $u(x,0) = e^{2x}$. $Ans: u(x,t) = e^{2x-t}$

5. $u_t - 2u_x = u^2$ with $u(x,0) = \sin x$. $Ans: u(x,t) = \dfrac{-1}{t - \csc(x+2t)}$