Applications of Partial Derivatives

1. Jacobian Matrix

The Jacobian matrix is a matrix made up of the first-order partial derivatives of a multivariable function $F(x,y) = \big(U(x,y),\ V(x,y)\big)$. Here's the formula for it:

$$\mathrm{J} = \begin{bmatrix} U_x & U_y \\ V_x & V_y \end{bmatrix}$$

The determinant of Jacobian matrix is denoted by $|\mathrm{J}| = U_x V_y - V_x U_y$

Example 1: Let $U = x^2 - y^2, V = 2xy$. Find Jacobian matrix and its determinant.

$$U_x = 2x, U_y = -2y, V_x = 2y \text{ and } V_y = 2x$$

$$\mathrm{J} = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

$$|\mathrm{J}| = 2x \times 2x - 2y \times (-2y) = 4x^2 + 4y^2$$

2. Hessian Matrix

The Hessian matrix, denoted by H, is a matrix that organizes the second-order partial derivatives of a function $f(x,y)$.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

The determinant of Hessian matrix is denoted by $|H| = f_{xx} f_{yy} - f_{xy} f_{yx}$

Example 2: Determine the Hessian matrix and calculate its determinant at the point $(1,0)$. For $f(x,y) = (x^2 + y^2)^2$.

$$f_x = 4x(x^2 + y^2) = 4x^3 + 4xy^2 \quad \Rightarrow \quad f_{xx} = 12x^2 + 4y^2 \quad \Rightarrow \quad f_{xx}(1,0) = 12$$

$$f_y = 4y(x^2 + y^2) = 4yx^2 + 4y^3 \quad \Rightarrow \quad f_{yy} = 4x^2 + 12y^2 \quad \Rightarrow \quad f_{yy}(1,0) = 4$$

$$f_{xy} = f_{yx} = 4y \times 2x = 8xy \quad \Rightarrow \quad f_{xy}(1,0) = 0$$

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12 & 0 \\ 0 & 4 \end{bmatrix}$$

$$|H| = \begin{vmatrix} 12 & 0 \\ 0 & 4 \end{vmatrix} = 48$$

3. Gradient of a Scalar Field

Let us first define the del operator such that:

$$\nabla = \frac{\partial}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial}{\partial z}\hat{\boldsymbol{k}}$$

A scalar field is a function that assigns a numerical value to every point in space, like temperature or pressure distributions. For example $f(x,y,z) = x^2 + \cos(2yz)$.

The Gradient of a scalar field $f(x,y,z)$ is a vector field denoted by ∇f defined as:

$$\nabla f = \frac{\partial f}{\partial x}\hat{\boldsymbol{i}} + \frac{\partial f}{\partial y}\hat{\boldsymbol{j}} + \frac{\partial f}{\partial z}\hat{\boldsymbol{k}}$$

or

$$\boxed{\nabla f = f_x\hat{\boldsymbol{i}} + f_y\hat{\boldsymbol{j}} + f_z\hat{\boldsymbol{k}}}$$

Example 3: Find ∇f for $f(x,y,z) = 2x^2 \sin y - xy \tan z$

$$f_x = 4x \sin y - y \tan z\,,\quad f_y = 2x^2 \cos y - x \tan z \text{ and } f_z = -xy \sec^2 z$$

$$\nabla f = (4x \sin y - y \tan z)\,\hat{\boldsymbol{i}} + (2x^2 \cos y - x \tan z)\,\hat{\boldsymbol{j}} - xy \sec^2 z\;\hat{\boldsymbol{k}}$$

4. Laplacian Operator

The differential operator ∇^2 is called Laplacian operator, defined as:

$$\nabla^2 = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

or

$$\boxed{\nabla^2 f = f_{xx} + f_{yy} + f_{zz}}$$

Example 4: Find $\nabla^2 f$ for $f(x,y,z) = x^3 e^y + xy^2 z^3$

$$f_x = 3x^2 e^y + y^2 z^3 \quad \Rightarrow \quad f_{xx} = 6xe^y$$

$$f_y = x^3 e^y + 2xyz^3 \quad \Rightarrow \quad f_{yy} = x^3 e^y + 2xz^3$$

$$f_z = 3xy^2 z^2 \quad \Rightarrow \quad f_{zz} = 6xy^2 z$$

$$\nabla^2 f = f_{xx} + f_{yy} + f_{zz} = 6xe^y + x^3 e^y + 2xz^3 + 6xy^2 z$$

5. Divergence of a Vector Field

The Divergence of a vector field $\vec{\boldsymbol{F}}(x,y,z) = F_1(x,y,z)\hat{\boldsymbol{i}} + F_2(x,y,z)\hat{\boldsymbol{j}} + F_3(x,y,z)\widehat{\boldsymbol{k}}$ is computed as:

$$\operatorname{div}\vec{\boldsymbol{F}} = \nabla\cdot\vec{\boldsymbol{F}} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Example 5: Find $div\ F$ if $\vec{\boldsymbol{F}}(x,y,z) = xz\hat{\boldsymbol{i}} + e^{yz}\hat{\boldsymbol{j}} - \ln(xy)\,\widehat{\boldsymbol{k}}$

$$\operatorname{div}\vec{\boldsymbol{F}} = \nabla\cdot\vec{\boldsymbol{F}} = \frac{\partial(xz)}{\partial x} + \frac{\partial(e^{yz})}{\partial y} - \frac{\partial(\ln(xy))}{\partial z} = z + ze^{yz}$$

6. The Curl of a Vector Field

The Curl of a vector field $\vec{\boldsymbol{F}}(x,y,z) = F_1(x,y,z)\hat{\boldsymbol{i}} + F_2(x,y,z)\hat{\boldsymbol{j}} + F_3(x,y,z)\widehat{\boldsymbol{k}}$ it is another vector defined as the following determinant:

$$curl\,\vec{\boldsymbol{F}} = \nabla\times\vec{\boldsymbol{F}} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \widehat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\hat{\boldsymbol{i}} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\widehat{\boldsymbol{k}}$$

Example 6: Find $curl\,\vec{\boldsymbol{F}}$ if $\vec{\boldsymbol{F}}(x,y,z) = xy\hat{\boldsymbol{i}} + yz\hat{\boldsymbol{j}} + xz\widehat{\boldsymbol{k}}$ at $(-1,-3,-2)$.

$$curl\,\vec{\boldsymbol{F}} = \nabla\times\vec{\boldsymbol{F}} = \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \widehat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & yz & xz \end{vmatrix}$$

$$= \left(\frac{\partial(xz)}{\partial y} - \frac{\partial(yz)}{\partial z}\right)\hat{\boldsymbol{i}} - \left(\frac{\partial(xz)}{\partial x} - \frac{\partial(xy)}{\partial z}\right)\hat{\boldsymbol{j}} + \left(\frac{\partial(yz)}{\partial x} - \frac{\partial(xy)}{\partial y}\right)\widehat{\boldsymbol{k}}$$

$$= -y\hat{\boldsymbol{i}} - z\hat{\boldsymbol{j}} - x\widehat{\boldsymbol{k}}$$

$$curl\,\vec{\boldsymbol{F}}\Big|_{\text{at }(-1,-3,-2)} = 3\hat{\boldsymbol{i}} + 2\hat{\boldsymbol{j}} + \widehat{\boldsymbol{k}}$$

H.W.

1. Let $U = x\cos y$, $V = x\sin y$. Find Jacobian matrix and its determinant at $(1,\pi/4)$.
2. Find the determinant of the Hessian matrix for $f(x,y) = x^2y + xy^2$.
3. If $f(x,y,z) = x^3y^2z$, then find ∇f and $\nabla^2 f$ at $(1,-1,2)$.
4. If $\vec{\boldsymbol{F}}(x,y,z) = yze^{xy}\hat{\boldsymbol{i}} + xze^{xy}\hat{\boldsymbol{j}} + (e^{xy} + 3\cos 3z)\widehat{\boldsymbol{k}}$, then find
   (a) $div\,\vec{\boldsymbol{F}}$ at $(0,\sqrt{6},\pi/6)$ (b) $curl\,\vec{\boldsymbol{F}}$