

## Sequences and Series

A sequence of real numbers is a function  $a : N \rightarrow R$ , denoted by  $\{a_n\}$ .

For example, the expression  $\{2n\}$  denotes the sequence  $\{2, 4, 6, \dots\}$

The number  $a_n$  is called the general term of the sequence  $\{a_n\}$ .

**Convergent sequence:** We say that the sequence  $\{a_n\}$  converges to  $L$  if  $\lim_{n \rightarrow \infty} a_n = L$ , otherwise it diverges.

**Example 1:** Determine whether the sequence converges or diverges.

$$1. \left\{ \frac{n^2 + 1}{(n + 1)^2} \right\} \quad 2. \left\{ \frac{n + 17}{\sqrt{2n^2 + 3n}} \right\}$$

**Solution:**

$$1. \lim_{n \rightarrow \infty} \frac{n^2 + 1}{(n + 1)^2} = \lim_{n \rightarrow \infty} \frac{2n}{2(n + 1)} = \lim_{n \rightarrow \infty} \frac{2}{2} = 1 \quad \text{converges to } 1$$

$$2. \lim_{n \rightarrow \infty} \frac{n + 17}{\sqrt{2n^2 + 3n}} = \lim_{n \rightarrow \infty} \frac{1 + \frac{17}{n}}{\sqrt{2 + \frac{3}{n}}} = \frac{1}{\sqrt{2}} \quad \text{converges to } \frac{1}{\sqrt{2}}$$

### • Limits that Arise Frequently

$$1. \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$$

$$2. \lim_{n \rightarrow \infty} \sqrt[n]{n} = 1$$

$$3. \lim_{n \rightarrow \infty} x^{\frac{1}{n}} = 1 \quad ; \quad (x > 0)$$

$$4. \lim_{n \rightarrow \infty} x^n = 0 \quad ; \quad (|x| < 1)$$

**Example 2:** Determine whether the sequence converges or diverges.

$$1. \left\{ \frac{1 + \ln n}{n} \right\} \quad 2. \left\{ \frac{1}{(0.6)^n} \right\}$$

**Solution:**

$$1. \lim_{n \rightarrow \infty} \frac{1 + \ln n}{n} = \lim_{n \rightarrow \infty} \frac{1}{n} + \lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0 \quad \text{converges to } 0$$

$$2. \lim_{n \rightarrow \infty} \frac{1}{(0.6)^n} = \frac{1}{\lim_{n \rightarrow \infty} (0.6)^n} = \infty \quad \text{diverges}$$

## Infinite Series

An infinite series is given by the terms of an infinite sequence, added together. For example, we could take the infinite sequence  $\{2n\} = \{2, 4, 6, \dots\}$ . Then the corresponding example of an infinite series would be given by all of these terms added together  $2 + 4 + 6 + \dots$ , so we have  $\sum_{n=1}^{\infty} 2n = 2 + 4 + 6 + \dots$

### • Geometric Series

A geometric series is any series that can be written in the form:

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \dots + ar^n + \dots$$

1. If  $|r| < 1$  then the geometric series converges to the sum  $S_n = \frac{a}{1-r}$
2. If  $|r| \geq 1$  then the geometric series diverges to  $\infty$ .

For example the series  $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

converges because  $r = \frac{1}{2} < 1$  and the sum is  $s_n = \frac{1/2}{1 - (1/2)} = 1$ .

The series  $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$  diverges to  $\infty$  because  $r = \frac{3}{2} > 1$

### • The $p$ – Series

If  $p$  is a real constant, then the series  $\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$

1. Converges if  $p > 1$ .
2. Diverges if  $p \leq 1$ .

For example the series  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  converges because  $p = 2 > 1$ .

The series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$  diverges to  $\infty$  because  $p = \frac{1}{2} < 1$ .

## • Tests for Converges of Series

### 1. Integral Test

Let the function  $f(x) = a_n(x)$  be continuous, positive and decreasing then the

series  $\sum_{n=1}^{\infty} a_n$  and the integral  $\int_1^{\infty} f(x) dx$  both converge or both diverge.

**Example 3:** Check series convergence. Explain

$$1. \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \qquad 2. \sum_{n=1}^{\infty} \frac{n}{n^2 + 1}$$

**Solution:**

$$1. \int_1^{\infty} \frac{dx}{x^2 + 1} = \tan^{-1} x \Big|_1^{\infty} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \text{converge}$$

$$2. \int_1^{\infty} \frac{x dx}{x^2 + 1} = \frac{1}{2} \ln|x^2 + 1| \Big|_1^{\infty} = \infty \quad \text{diverge}$$

### 2. Ratio Test

Let  $\sum_{n=1}^{\infty} a_n$  be a series with non – negative terms and  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = L$ .

If 1.  $L < 1$  then the series converges

2.  $L > 1$  then the series diverges

3.  $L = 1$  then this test is inconclusive

**Example 3:** Check series convergence. Explain

$$1. \sum_{n=1}^{\infty} \frac{n^3}{3^n} \qquad 2. \sum_{n=1}^{\infty} \frac{n!}{2^n} \qquad 3. \sum_{n=1}^{\infty} \frac{3}{2n + 5}$$

**Solution:**

$$1. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)^3}{3^{n+1}} \times \frac{3^n}{n^3} = \frac{1}{3} \lim_{n \rightarrow \infty} \left( \frac{n+1}{n} \right)^3$$

$$= \frac{1}{3} \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^3 = \frac{1}{3} < 1 \quad \text{converges}$$

$$2. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(n+1)!}{2^{n+1}} \times \frac{2^n}{n!}$$

$$= \lim_{n \rightarrow \infty} \frac{(n+1) \times n!}{2^n \times 2} \times \frac{2^n}{n!} = \lim_{n \rightarrow \infty} \frac{n+1}{2} = \infty > 1 \quad \text{diverges}$$

$$3. \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{3}{2(n+1)+5} \times \frac{2n+5}{3}$$

$$= \lim_{n \rightarrow \infty} \frac{2n+5}{2n+7} = 1. \quad \text{The test is inconclusive.}$$

So we try another test  $\int_1^{\infty} \frac{3dx}{2x+5} = \frac{3}{2} \ln|2x+5| \Big|_1^{\infty} = \infty \quad \text{diverges}$

### 3. Root Test

Let  $\sum_{n=1}^{\infty} a_n$  be a series with non – negative terms and  $\lim_{n \rightarrow \infty} \sqrt[n]{a_n} = L$ . Then,

1. when  $L < 1$ , the series converges

2. when  $L > 1$ , the series diverges

3. when  $L = 1$ , this test is inconclusive

**Example 4:** Check series convergence. Explain

$$1. \sum_{n=1}^{\infty} \left(\frac{3n+2}{4n+1}\right)^n \quad 2. \sum_{n=1}^{\infty} \left(\frac{1}{n}\right)^n \quad 3. \sum_{n=1}^{\infty} \frac{2^n}{n^2}$$

**Solution:**

$$1. \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{3n+2}{4n+1} = \frac{3}{4} < 1 \quad \text{converges}$$

$$2. \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{1}{n} = 0 < 1 \quad \text{converges}$$

$$3. \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{2}{\sqrt[n]{n^2}} = \frac{2}{\lim_{n \rightarrow \infty} \sqrt[n]{n}} = \frac{2}{1} = 2 > 1 \quad \text{diverges}$$

## H.W

1. Determine whether the sequence converges or diverges.

a.  $\left\{ \frac{3 + \ln n^n}{n^2} \right\}$

b.  $\left\{ \left( \frac{3}{n} \right)^{\frac{1}{n}} \right\}$

c.  $\left\{ \left( \frac{n+5}{n} \right)^n \right\}$

2. Find the sum of the following series:

a.  $\sum_{n=1}^{\infty} \frac{2^n}{5^n}$

b.  $\sum_{n=1}^{\infty} \frac{5^n}{3^{2n}}$

3. Check series convergence. Explain

a.  $\sum_{n=1}^{\infty} \frac{1}{n^2 \sqrt{n}}$

b.  $\sum_{n=1}^{\infty} \frac{5^n}{n 3^n}$

c.  $\sum_{n=1}^{\infty} \left( \frac{n}{2n-1} \right)^n$