Partial Derivatives

For a function of two independent variables, $z = f(x,y)$, the partial derivative of f or z with respect to x represented by $f_x, z_x, \partial f/\partial x$ or $\partial z/\partial x$, we can be found f_x by applying all the usual rules of differentiation. The only exception is that, whenever and wherever the second variable y appears, it is treated as a constant in every respect. The partial derivative of f or z with respect to y represented by $f_y, z_y, \partial f/\partial y$ or $\partial z/\partial y$, we can similarly be found by treating x as a constant whenever it appears.

For a function of more than two independent variables, the process of finding the partial derivative of a function is called partial differentiation. In this process, the partial derivative of a function with respect to one variable is found by keeping the other variable constant.

Example 1: Find f_x and f_y for the function

1. $f(x,y) = x^2y^4 \quad \Rightarrow \quad f_x = 2xy^4$ and $f_y = 4x^2y^3$
2. $f(x,y) = x^3 + y^2 \quad \Rightarrow \quad f_x = 3x^2$ and $f_y = 2y$
3. $f(x,y) = e^{2y+3}\sin 3x \quad \Rightarrow \quad f_x = 3e^{2y+3}\cos 3x$ and $f_y = 2e^{2y+3}\sin 3x$

Example 2: Find w_x and w_y for $w(x,y) = x^2\cos(xy)$

$$w_x = x^2(-\sin(xy)) \times y + 2x\ \cos(xy) = -x^2y\ \sin(xy) + 2x\ \cos(xy)$$

$$w_y = x^2(-\sin(xy)) \times x = -x^3\ \sin(xy)$$

Example 3: Find h_s and h_t for $h(s,t) = t\ln(4s^2+1) + t^2\tan^{-1}(2s)$

$$h_s = \frac{8st}{4s^2+1} + \frac{2t^2}{1+4s^2} = \frac{8st+2t^2}{4s^2+1}$$

$$h_t = \ln(4s^2+1) + 2t\tan^{-1}(2s)$$

Example 4: If $f(x,y) = \dfrac{x-y}{x+y}$, then show that $xf_x + yf_y = 0$

$$f_x = \frac{(x+y)-(x-y)}{(x+y)^2} = \frac{2y}{(x+y)^2}$$

$$f_y = \frac{(x+y)\times(-1)-(x-y)}{(x+y)^2} = \frac{-2x}{(x+y)^2}$$

$$xf_x + yf_y = \frac{2xy}{(x+y)^2} + \frac{-2xy}{(x+y)^2} = 0$$

Example 5: If $w = x\sin(yz) + xe^{yz}$, then show that 1. $xw_x = w$ 2. $yw_y = zw_z$

1. $w_x = \sin(yz) + e^{yz}$ ⇨ $xw_x = x\sin(yz) + xe^{yz} = w$
2. $w_y = xz\cos(yz) + xze^{yz}$ ⇨ $yw_y = xyz\cos(yz) + xyze^{yz}$

   $w_z = xy\cos(yz) + xye^{yz}$ ⇨ $zw_z = xyz\cos(yz) + xyze^{yz}$

So, $yw_y = zw_z$

Second Order Partial Derivatives

Let $z = f(x,y)$ be a function of x and y, then the second partial derivative of f with respect to x is f_{xx} , the second partial derivative of f with respect to y is f_{yy} , the second partial derivative of f with respect to y and then with respect to x is f_{xy} and the second partial derivative of f with respect to x and then with respect to y is f_{yx}.

$$\left(f_{xx} \equiv \frac{\partial^2 f}{\partial x^2}, f_{yy} \equiv \frac{\partial^2 f}{\partial y^2}, f_{xy} \equiv \frac{\partial^2 f}{\partial x \partial y} \text{ and } f_{yx} \equiv \frac{\partial^2 f}{\partial y \partial x}\right)$$

f_{xy} and f_{yx} are called mixed partial derivatives where $f_{xy} = f_{yx}$.

Example 6: Find f_{xx} , f_{xy} , f_{yx} and f_{yy} for $f(x,y) = x^2 + 3xy - y^4$

$f_x = 2x + 3y$ ⇨ $f_{xx} = 2$ and $f_{yx} = 3$

$f_y = 3x - 4y^3$ ⇨ $f_{yy} = -12y^2$ and $f_{xy} = 3$

Example 7: Find f_{rr} and $f_{\theta\theta}$ for $f(r,\theta) = r^2\sin^2\theta$

$f_r = 2r\sin^2\theta$ ⇨ $f_{rr} = 2\sin^2\theta$

$f_\theta = 2r^2\sin\theta\cos\theta$ but $(2\sin\theta\cos\theta = \sin 2\theta)$

$f_\theta = r^2\sin 2\theta$ ⇨ $f_{\theta\theta} = 2r^2\cos 2\theta$

Example 8: Find all first and second order partial derivatives of the function

$f(x,y,z) = 3x^2 - 2xy^2 + 4x^2z + z^3y + 5$

$f_x = 6x - 2y^2 + 8xz$ ⇨ $f_{xx} = 6 + 8z$, $f_{yx} = -4y$ and $f_{zx} = 8x$

$f_y = -4xy + z^3$ ⇨ $f_{yy} = -4x$, $f_{xy} = -4y$ and $f_{zy} = 2z^2$

$f_z = 4x^2 + 2z^2y$ ⇨ $f_{zz} = 4zy$, $f_{xz} = 8x$ and $f_{yz} = 2z^2$

Chain Rule for Partial Derivatives

If $z = f(x, y)$ and $x = x(u, v)$, $y = y(u, v)$, then $z = z(u, v)$ and

$$\boxed{\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}} \quad \text{and} \quad \boxed{\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}}$$

Example 9: If $z = x^2 + y^2, x = 2u + v , y = 2v - u.$

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ as a functions of u and v

$$\frac{\partial z}{\partial x} = 2x = 4u + 2v \,, \qquad \frac{\partial z}{\partial y} = 2y = 4v - 2u$$

$$\frac{\partial x}{\partial u} = 2 \,, \quad \frac{\partial y}{\partial u} = -1, \quad \frac{\partial x}{\partial v} = 1 \quad \text{and} \quad \frac{\partial y}{\partial v} = 2$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial u}$$

$$= (4u + 2v) \times 2 + (4v - 2u) \times (-1)$$

$$= 8u + 4v - 4v + 2u = 10u$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \times \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \times \frac{\partial y}{\partial v}$$

$$= (4u + 2v) \times 1 + (4v - 2u) \times 2$$

$$= 4u + 2v + 8v - 4u = 10v$$

Example 10: Let $w = xy - z, \quad x = \sin t \,, \quad y = \cos t$ and $z = t$. Find $\frac{\partial w}{\partial t}$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \times \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \times \frac{\partial y}{\partial t} + \frac{\partial w}{\partial z} \times \frac{\partial z}{\partial t}$$

$$= y \times \cos t + x \times (-\sin t) + (-1) \times 1$$

$$= \cos^2 t - \sin^2 t - 1$$

Laplace's Equation: We say that the function $f(x, y)$ satisfies Laplace's equation if

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0} \qquad (f_{xx} + f_{yy} = 0)$$

Example 11: Show that the functions satisfy Laplace's equation

1. $f(x, y) = e^{-2y} \cos 2x$

2. $w(s, t) = \ln(t^2 + s^2)$

1. $\frac{\partial f}{\partial x} = -2e^{-2y}\sin 2x \quad \Rightarrow \quad \frac{\partial^2 f}{\partial x^2} = -4e^{-2y}\cos 2x$

$\frac{\partial f}{\partial y} = -2e^{-2y}\cos 2x \quad \Rightarrow \quad \frac{\partial^2 f}{\partial y^2} = 4e^{-2y}\cos 2x$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = -4e^{-2y}\cos 2x + 4e^{-2y}\cos 2x = 0$$

2. $\frac{\partial w}{\partial s} = \frac{2s}{t^2+s^2} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial s^2} = \frac{2(t^2+s^2) - 2s\times 2s}{(t^2+s^2)^2}$

$$\frac{\partial^2 w}{\partial s^2} = \frac{2t^2+2s^2-4s^2}{(t^2+s^2)^2} = \frac{2t^2-2s^2}{(t^2+s^2)^2}$$

$\frac{\partial w}{\partial t} = \frac{2t}{t^2+s^2} \quad \Rightarrow \quad \frac{\partial^2 w}{\partial t^2} = \frac{2(t^2+s^2) - 2t\times 2t}{(t^2+s^2)^2}$

$$\frac{\partial^2 w}{\partial t^2} = \frac{2t^2+2s^2-4t^2}{(t^2+s^2)^2} = \frac{2s^2-2t^2}{(t^2+s^2)^2}$$

$$\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial t^2} = \frac{2t^2-2s^2}{(t^2+s^2)^2} + \frac{2s^2-2t^2}{(t^2+s^2)^2} = 0$$

The 1-D Heat Equation: The 1-D Heat equation takes the form:

$$\boxed{\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}}, \quad \text{where } k > 0 \text{ which is called the thermal diffusivity}$$

Example 12: Show that the function $T(x,t) = 3e^{-4\pi^2 t}\cos(2\pi x)$ satisfy Heat quation, with $k = 1$.

$$\frac{\partial T}{\partial t} = -12\pi^2 e^{-4\pi^2 t}\cos(2\pi x)$$

$\frac{\partial T}{\partial x} = -6\pi e^{-4\pi^2 t}\sin(2\pi x) \quad \Rightarrow \quad \frac{\partial^2 T}{\partial x^2} = -12\pi^2 e^{-4\pi^2 t}\cos(2\pi x)$

$$\therefore \frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

Example 13: If $T(x,t) = 2e^{-12t}\sin 2x$ satisfy Heat equation, then find the thermal diffusivity k.

$$\frac{\partial T}{\partial t} = -24e^{-12t}\sin 2x$$

$$\frac{\partial T}{\partial x} = 4e^{-12t}\cos 2x \quad \Rightarrow \quad \frac{\partial^2 T}{\partial x^2} = -8\, e^{-12t}\sin 2x$$

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \quad \Rightarrow \quad -24e^{-12t}\sin 2x = -8ke^{-12t}\sin 2x$$

$$\therefore \quad k = 3$$

Wave Equation: The wave equation takes the form: $\boxed{u_{tt} = c^2 u_{xx}}$

Example 14: Show that the function $u(x,t) = \cos(x+2t) - \cos(x-2t)$ satisfy the wave equation $u_{tt} = 4u_{xx}$

$$u_t = -2\sin(x+2t) - 2\sin(x-2t)$$

$$u_{tt} = -4\cos(x+2t) + 4\cos(x-2t)$$

$$u_x = -\sin(x+2t) - \sin(x-2t)$$

$$u_{xx} = -\cos(x+2t) + \cos(x-2t)$$

$$\therefore\ u_{tt} = 4u_{xx}$$

Exercises

1. If $w = \cos(x+y) + \sin(x-y)$ then show that $\dfrac{\partial^2 w}{\partial x^2} = \dfrac{\partial^2 w}{\partial y^2}$.

2. Find $\dfrac{\partial^2 z}{\partial x \partial y}$ if $z = x^2\sin(2x-3y)$.

3. If $z = \ln(xy)$ and $x = r\sin\theta$, $y = r\cos\theta$ then show that $\dfrac{\partial z}{\partial \theta} = 2\cot 2\theta$.

4. If $w = (x^2+y^2+z^2)\,(y^2+z^2)$, then find $\dfrac{\partial w}{\partial x}$ at $x = 1, y = 2$ and $z = 3$.

5. Show that the functions satisfy Laplace's equation

   $a)\ f(x,y) = e^{3x}\sin 3y$ $\qquad$ $b) f(x,y) = x^3 - 3xy^2$

6. If $T(x,t) = 5e^{-32\pi^2 t}\sin(4\pi x)$ satisfy Heat equation, then find the thermal diffusivity k.

7. Show that the function $u(x,t) = e^{x+ct} - e^{x-ct}$ satisfy the wave equation.