Beta Function

Beta function is defined by the integral

$$B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx;\quad n>0, m>0$$

Beta function satisfies the recursive properties:

1. The Beta function is symmetric that is : $B(m,n)=B(n,m)$

2. $B(m,n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

3. $\displaystyle\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta=\frac{1}{2}B(m,n)$

Example 1: Evaluate 1. $B(3,4)$ 2. $B\left(\frac{1}{2},\frac{5}{2}\right)$

$$1.\ B(3,4)=\frac{\Gamma(3)\Gamma(4)}{\Gamma(7)}=\frac{2!\times 3!}{6!}$$

$$=\frac{2\times 3!}{6\times 5\times 4\times 3!}=\frac{1}{60}$$

$$2.\ B\left(\frac{1}{2},\frac{5}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{5}{2}\right)}$$

$$=\frac{\sqrt{\pi}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}}{\Gamma(3)}=\frac{3\pi}{8}$$

Example 2: Evaluate each of the following integrals

$$1.\ \int_0^1 x^3(1-x)^4\,dx=B(4,5)=\frac{\Gamma(4)\Gamma(5)}{\Gamma(9)}=\frac{3!\times 4!}{8!}$$

$$=\frac{6\times 4!}{8\times 7\times 6\times 5\times 4!}=\frac{1}{280}$$

2. $\int_0^2 \frac{x^2}{\sqrt{2-x}}dx$

Let $x = 2y \quad \Rightarrow \quad dx = 2dy$

$x = 0 \quad \Rightarrow \quad y = 0 \quad$ and $\quad x = 2 \quad \Rightarrow \quad y = 1$

$$\int_0^2 \frac{x^2}{\sqrt{2-x}}dx = \int_0^1 \frac{4y^2}{\sqrt{2-2y}} \times 2dy = \frac{8}{\sqrt{2}}\int_0^1 \frac{y^2}{\sqrt{1-y}}dy$$

$$= 4\sqrt{2}\int_0^1 y^2(1-y)^{-1/2}\,dy$$

$$= 4\sqrt{2}\,B\left(3,\frac{1}{2}\right) = 4\sqrt{2}\,\frac{\Gamma(3)\Gamma(1/2)}{\Gamma(7/2)} = \frac{64\sqrt{2}}{15}$$

3. $\int_0^{\pi/2} \sin^9\theta\cos^5\theta\,d\theta$

$2m - 1 = 9 \;\Rightarrow\; m = 5 \quad$ and $\quad 2n - 1 = 5 \;\Rightarrow\; n = 3$

$$\int_0^{\pi/2} \sin^9\theta\cos^5\theta\,d\theta = \frac{1}{2}B(5,3) = \frac{\Gamma(5)\Gamma(3)}{2\Gamma(5+3)} = \frac{4!\times 2!}{2\times 7!} = \frac{1}{210}$$

4. $\int_0^{\pi/2} \sin^5 x\,dx \;;\; 2m - 1 = 5 \;\Rightarrow\; m = 3 \quad$ and $\quad 2n - 1 = 0 \;\Rightarrow\; n = \frac{1}{2}$

$$\int_0^{\pi/2} \sin^5 x\,dx = \frac{1}{2}B\left(3,\frac{1}{2}\right) = \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}\right)} = \frac{2!\sqrt{\pi}}{2\times\frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\sqrt{\pi}} = \frac{8}{15}$$

Exercises

Evaluate each of the following integrals

1. $\int_0^1 x^5(1-x)^6\,dx$ 2. $\int_0^{\pi/2} \cos^4 x\,dx$ 3. $\int_0^{\pi/2} \sin^5 x\cos^4 x\,dx$ 4. $\int_0^{\pi/2} \sqrt{\tan x}\,dx$