Integral

If $F(x)$ is an antiderivative of $f(x)$, then the expression $F(x) + C$ where C is an arbitrary constant, is called the indefinite integral of $f(x)$.

Integral Calculus Formula Sheet

No.	Derivation Formulas	Integration Formulas
1.	$\frac{d}{dx}(C) = 0$	$\int 0.\, dx = C$
2.	$\frac{d}{dx}(x^n) = nx^{n-1}$	$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C$
3.	$\frac{d}{dx}(\sin ax) = a\cos ax$	$\int \cos ax\, dx = (1/a)\sin ax + C$
4.	$\frac{d}{dx}(\cos ax) = -a\sin ax$	$\int \sin ax\, dx = -(1/a)\cos ax + C$
5.	$\frac{d}{dx}(\tan ax) = a\sec^2 ax$	$\int \sec^2 ax\, dx = (1/a)\tan ax + C$
6.	$\frac{d}{dx}(\cot ax) = -a\csc^2 ax$	$\int \csc^2 ax\, dx = -(1/a)\cot ax + C$
7.	$\frac{d}{dx}(\sec ax) = a\sec ax\tan ax$	$\int \sec ax\tan ax\, dx = (1/a)\sec ax + C$
8.	$\frac{d}{dx}(\csc ax) = -a\csc ax\cot ax$	$\int \csc ax\cot ax\, dx = -(1/a)\csc ax + C$
9.	$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$	$\int \frac{1}{\sqrt{1-x^2}}dx = \sin^{-1} x + C$
10.	$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$	$\int \frac{1}{1+x^2}dx = \tan^{-1} x + C$
11.	$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{x\sqrt{x^2-1}}$	$\int \frac{1}{x\sqrt{x^2-1}}dx = \sec^{-1} x + C$
12.	$\frac{d}{dx}(\ln x) = \frac{1}{x}$	$\int \frac{1}{x}dx = \ln\lvert x\rvert + C$
13.	$\frac{d}{dx}(e^{ax}) = ae^{ax}$	$\int e^{ax}\, dx = (1/a)e^{ax} + C$

The Integrals Concerning the Inverse Trigonometric Functions

14.	$\int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c_1 = -\cos^{-1}\frac{u}{a} + c_2$
15.	$\int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c_1 = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c_2$
16.	$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{u}{a} + c_1 = -\frac{1}{a}\csc^{-1}\frac{u}{a} + c_2$

Examples

1. $\int x\sqrt{2x}\,dx = \sqrt{2}\int (x)^{\frac{3}{2}}\,dx = \sqrt{2}\,(x)^{\frac{5}{2}} \times \frac{2}{5} + c = \frac{2\sqrt{2}}{5}(x)^{\frac{5}{2}} + c$

2. $\int \sqrt{3x+1}\,dx = \frac{1}{3}\int 3\,(3x+1)^{\frac{1}{2}}dx = \frac{1}{3}(3x+1)^{\frac{3}{2}} \times \frac{2}{3} + c = \frac{2}{9}(3x+1)^{\frac{3}{2}} + c$

3. $\int \frac{dx}{(2x-3)^2} = \frac{1}{2}\int 2\,(2x-3)^{-2}dx = -\frac{1}{2}(2x-3)^{-1} + c = -\frac{1}{2(2x-3)} + c$

4. $\int \frac{xdx}{\sqrt{x^2-3}} = \frac{1}{2}\int 2x(x^2-3)^{-1/2}dx = \frac{1}{2}(x^2-3)^{1/2} \times 2 + c = \sqrt{x^2-3} + c$

5. $\int \frac{xdx}{x^2-3} = \frac{1}{2}\ln|x^2-3| + c$

6. $\int \sqrt{1+\cos 2x}\,dx = \int \sqrt{2\cos^2 x}\,dx = \int \sqrt{2}\,\cos x\,dx = \sqrt{2}\sin x + c$

7. $\int \frac{\sin 2x}{\sqrt{1+\cos 2x}}dx = \frac{-1}{2}\int (1+\cos 2x)^{-1/2}(-2\sin 2x)\,dx = -\sqrt{1+\cos 2x} + c$

8. $\int \frac{\sin 2x}{1+\cos 2x}dx = \frac{-1}{2}\int \frac{-2\sin 2x}{1+\cos 2x}dx = \frac{-1}{2}\ln|1+\cos 2x| + c$

9. $\int (2e^{-2x} + 3x^2)\,dx = -e^{-2x} + x^3 + c$

10. $\int \tan 2x\,dx = \int \frac{\sin 2x}{\cos 2x}dx = -\frac{1}{2}\ln|\cos 2x| + c$

11. $\int \frac{dx}{\sqrt{2-x^2}} = \sin^{-1}\frac{x}{\sqrt{2}} + c$

12. $$\int \frac{dx}{x^2+2x+2} = \int \frac{dx}{x^2+2x+1+1}$$

$$= \int \frac{dx}{(x+1)^2+1} = \tan^{-1}(x+1)+c$$

13. $$\int \frac{dx}{x\sqrt{x^2-3}} = \frac{1}{\sqrt{3}}\sec^{-1}\frac{x}{\sqrt{3}}+c$$

14. $$\int \frac{2x}{x^2-6x+10}\,dx = \int \frac{2x-6+6}{x^2-6x+10}\,dx$$

$$= \int \frac{2x-6}{x^2-6x+10}\,dx + \int \frac{6}{x^2-6x+9+1}\,dx$$

$$= \ln|x^2-6x+10| + \int \frac{6}{(x-3)^2+1}\,dx$$

$$= \ln|x^2-6x+10| + 6\tan^{-1}(x-3)+c$$

Definite Integration

If $\int f(x)\,dx = F(x)+c$, then the definite integral of $f(x)$ on an interval $[a,b]$ is expressed as: $\int_a^b f(x)\,dx = F(b)-F(a).$

Examples

1. $$\int_0^3 x\sqrt{9-x^2}\,dx = -\frac{1}{2}\int_0^3 -2x(9-x^2)^{1/2}\,dx = -\frac{1}{2}(9-x^2)^{3/2}\times\frac{2}{3}\Big|_0^3$$

$$= -\frac{1}{3}\left((9-3^2)^{3/2}-(9-0^2)^{3/2}\right) = 9$$

2. $$\int_0^{\pi/2} \sin 2x\,dx = -\frac{1}{2}\cos 2x\Big|_0^{\pi/2} = -\frac{1}{2}(\cos\pi-\cos 0) = -\frac{1}{2}(-1-1) = 1$$

3. $$\int_0^{2/3} \frac{1}{\sqrt{4-9x^2}}\,dx = \frac{1}{3}\sin^{-1}\frac{3x}{2}\Big|_0^{2/3} = \frac{1}{3}(\sin^{-1}1-\sin^{-1}0) = \frac{\pi}{6}$$

$$4.\int_1^4 \frac{1}{x^2-2x+10}dx = \int_1^4 \frac{1}{x^2-2x+1+9}dx = \int_1^4 \frac{1}{(x-1)^2+9}dx = \frac{1}{3}\tan^{-1}\frac{x-1}{3}\Big|_1^4$$

$$= \frac{1}{3}\left(\tan^{-1}\frac{4-1}{3} - \tan^{-1}0\right) = \frac{1}{3}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{12}$$

Exercises

Evaluate the following integrals

1. $\int (x+\sqrt{x})dx$ 2. $\int \sqrt[3]{3x-2}\,dx$ 3. $\int \frac{xdx}{\sqrt{1-4x^2}}$

4. $\int \frac{xdx}{x^2+3}$ 5. $\int \frac{dx}{\sqrt{1-4x^2}}$ 6. $\int_0^{\sqrt{3}} \frac{dx}{x^2+3}$

7. $\int \sqrt{1-\cos 2x}\,dx$ 8. $\int \frac{(x-1)^2}{x\sqrt{x}}dx$ 9. $\int \frac{dx}{x^2-2x+5}$

Ans: 1. $\frac{x^2}{2}+\frac{2x^{3/2}}{3}+c$ 2. $\frac{(3x-2)^{4/3}}{4}+c$ 3. $-\frac{\sqrt{1-4x^2}}{4}+c$

4. $\frac{\ln(x^2+3)}{2}+c$ 5. $\frac{\sin^{-1}2x}{2}+c$ 6. $\frac{\pi}{4\sqrt{3}}$

7. $-\sqrt{2}\cos x+c$ 8. $\frac{2x^{3/2}}{3}-4\sqrt{x}-\frac{2}{\sqrt{x}}+c$ 9. $\frac{1}{2}\tan^{-1}\left(\frac{x-1}{2}\right)+c$