Differential Equations

A differential equation (DE) is an equation that involves one or more derivatives. Differential equations are classified by:

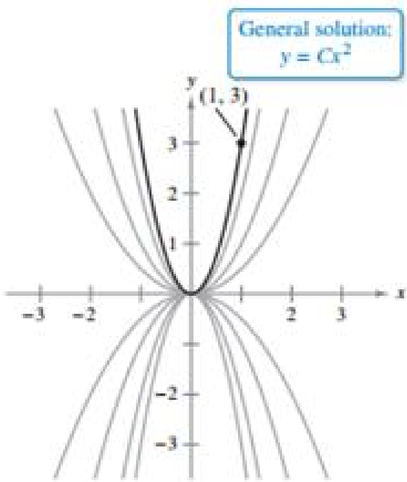
1. Type (Ordinary Differential Equation (ODE) or Partial Differential Equation (PDE))
2. Order (that of the highest order derivative that occurs in the equation)
3. Degree (the exponent of the highest order derivative)

For examples

1. $y' = \sin 2x + 3,$ is an ODE, of order one and degree one.
2. $y''' - 2y'' + y' = 2e^x,$ is an ODE, of order three and degree one.
3. $\left(\frac{d^3u}{dt^3}\right)^2 + \left(\frac{d^2u}{dt^2}\right)^4 + t^2u = \sin 3t\,,$ is an ODE, of order three and degree two.
4. $\frac{\partial^2 w}{\partial t^2} = a^2\frac{\partial^2 w}{\partial x^2},$ is a PDE, of order two and degree one.

General Solution and Particular Solution

A particular solution of a differential equation is any solution that is obtained by assigning specific values to the arbitrary constants in the general solution. Geometrically, the general solution of a differential equation represents a family of curves known as solution curves. For instance, the general solution of the differential equation $xy' - 2y = 0$ is $y = Cx^2$ (General solution)



Solution Curves for $xy' - 2y = 0$

The figure above shows several solution curves corresponding to different values of C.

Particular solutions of a differential equation are obtained from initial conditions placed on the unknown function and its derivatives. For instance, suppose you want to find the particular solution whose graph passes through the point $(1,3)$.

This initial condition can be written as $y = 3$ when $x = 1$ (Initial condition)

Substituting these values into the general solution produces $3 = C(1)^2$ which implies that $C = 3$. So, the particular solution is $y = 3x^2$ (Particular solution).

Direct Integration

An ordinary differential equation of the following form

$$\frac{d^n y}{dx^n} = f(x)$$

can be integrated directly by finding antiderivatives n-times.

Example 1: Find the general solution of ODE $y' = 2x + 5$.

Solution:

$$\frac{dy}{dx} = 2x + 5 \Rightarrow y = \int (2x+5)dx$$

$$y = x^2 + 5x + C \qquad \text{(General solution)}$$

Example 2: Find the general solution of ODE $y'' = -\sin 2x$.

Solution:

$$y' = \int -\sin 2x \, dx = \frac{1}{2}\cos 2x + C$$

$$y = \int \left(\frac{1}{2}\cos 2x + C\right) dx = \frac{1}{4}\sin 2x + Cx + K$$

Example 3: Find the general solution of the differential equation $y' = \sec^2 x$.

Then find the particular solution determined by the initial condition $y(\pi/4) = 3$.

Solution:

$$\frac{dy}{dx} = \sec^2 x \Rightarrow dy = \sec^2 x \, dx$$

$$y = \tan x + C \qquad \text{(General solution)}$$

$$y(\pi/4) = 3 \Rightarrow 3 = \tan(\pi/4) + C$$

$$3 = 1 + C \Rightarrow C = 2$$

$$y = \tan x + 2 \qquad \text{(Particular solution)}$$

Example 4: Find the particular solution of the differential equation $y'' = 6x$ determined by the initial conditions $y'(0) = 2$ and $y(1) = 3$

Solution: $y'' = 6x \quad \Rightarrow \quad y' = \int 6x\,dx = 3x^2 + C$

$$y'(0) = 2 \quad \Rightarrow \quad 2 = 3 \times 0 + C \quad \Rightarrow \quad C = 2$$

$$y' = 3x^2 + 2 \quad \Rightarrow \quad y = \int (3x^2 + 2)\,dx = x^3 + 2x + K$$

$$y(1) = 3 \quad \Rightarrow \quad 3 = (1)^3 + 2(1) + K \quad \Rightarrow \quad K = 0$$

$y = x^3 + 2x$ (Particular solution)

Example 5: Solve $y'' = xe^{2x}$ with $y'(0) = 1$, $y(0) = 7$

Solution:

$$y' = \int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + C$$

$$y'(0) = 1 \Rightarrow 1 = 0 - \frac{1}{4} + C \quad \Rightarrow C = \frac{5}{4}$$

$$y' = \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + \frac{5}{4}$$

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x	+	e^{2x}
1	−	$(1/2)e^{2x}$
0		$(1/4)e^{2x}$

$$y = \frac{1}{2}\int xe^{2x}\,dx + \frac{1}{4}\int (-e^{2x} + 5)\,dx$$

$$y = \frac{1}{2}\left(\frac{xe^{2x}}{2} - \frac{e^{2x}}{4}\right) + \frac{1}{4}\left(-\frac{e^{2x}}{2} + 5x\right) + K$$

$$y = \frac{e^{2x}(x-1) + 5x}{4} + K; \qquad y(0) = 7 \Rightarrow K = \frac{29}{4}$$

$$y = \frac{e^{2x}(x-1) + 5x + 29}{4}$$

Example 6: The velocity of an object is given by the equation $v = \sqrt{t+4}\ m/sec$. Find the distance traveled by the object for the first 5 sec.

Solution: We have $v = \frac{dx}{dt} \Rightarrow x = \int_0^5 \sqrt{t+4}\,dt = \int_0^5 (t+4)^{1/2}\,dt$

$$=\frac{2}{3}(t+4)^{3/2}\Big|_0^5=\frac{2}{3}(27-8)=\frac{38}{3}\ m.$$

Example 7: A particle starts from rest with an acceleration $a=\cos\frac{\pi t}{6}\ m/sec^2$.

Find the distance traveled by the particle for the 3rd second.

Solution: We have $a=\frac{dv}{dt}$ ⇨ $v=\int\cos\frac{\pi t}{6}dt=\frac{6}{\pi}\sin\frac{\pi t}{6}+C$

$v=0$ when $t=0$ ⇨ $C=0$

$$v=\frac{dx}{dt}=\frac{6}{\pi}\sin\frac{\pi t}{6}$$

$$x=\int_2^3\frac{6}{\pi}\sin\frac{\pi t}{6}dt=\frac{36}{\pi^2}\left(-\cos\frac{\pi t}{6}\right)\Big|_2^3=\frac{36}{\pi^2}\left(-\cos\frac{\pi}{2}+\cos\frac{\pi}{3}\right)$$

$$=\frac{36}{\pi^2}\left(0+\frac{1}{2}\right)=\frac{18}{\pi^2}\ m.$$

Exercises

1. Solve $y'=\tan x$ with $y(\pi/4)=3$.
2. Solve $y''=-x\sin x$ with $y'(0)=-3$, $y(0)=1$.
3. Find the particular solution for the differential equation $y''=\sin x+2$ determined by the initial conditions $y'(0)=1$, $y(0)=3$.
4. The velocity of an object is given by the equation $v=6-2t\ m/sec$. Find the distance traveled by the object for the first 3 sec.
5. A particle starts from rest with an acceleration $a=1+\cos(\pi t)\ m/sec^2$. Find the distance traveled by the particle for the first second.

Websites:

1. http://sites.science.oregonstate.edu/math/home/programs/undergrad/CalculusQuestStudyGuides/ode/first/di/di.html
2. https://math24.net/distance-velocity-acceleration.html#example1