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Chapter Two: Vectors and their properties

2.0 VECTORS

2.1 Introduction

If a sack of flour has a mass of 10kg, that mass is not dependent on where the flour, whether it at rest in a storeroom on land or in motion on a ship in sea. The above statement describes only the magnitude / size (10kg) but not the position. This shows that mass is a *scalar* quantity.

For a quantity like velocity it is quite different. To a passenger in Mombasa desiring to go to Nairobi city on a bus moving at 20m/s, it obviously makes a big difference whether the bus is moving towards Nairobi city or Malindi town. Here both direction and size/magnitude are vitally important. Such a quantity like velocity is a *vector* quantity.

2.2 Scalar and Vector quantities

Scalar quantity: It is a physical quantity that has no direction and it is completely specified by its magnitude / size alone, e.g. mass, energy, time, etc.

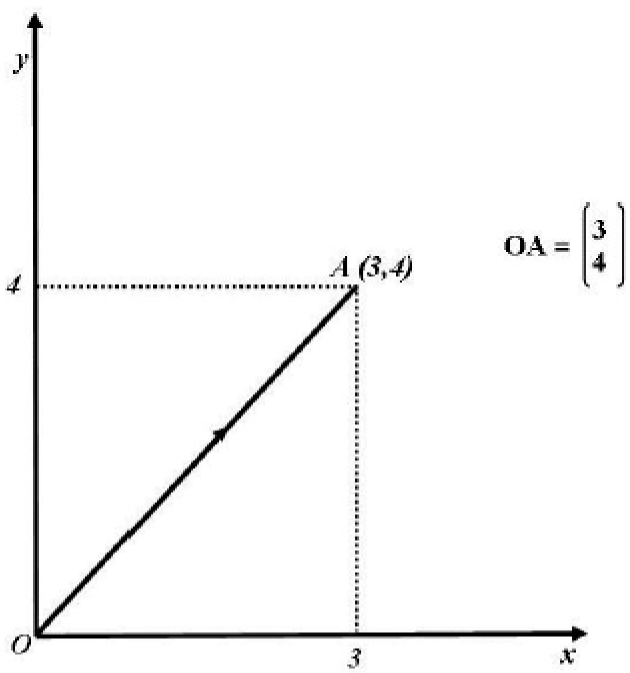
Vector quantity: It is a physical quantity that is completely specified only when both its magnitude / size and direction are given, e.g. velocity, displacement, force, momentum, acceleration etc.

2.3 Representing vectors

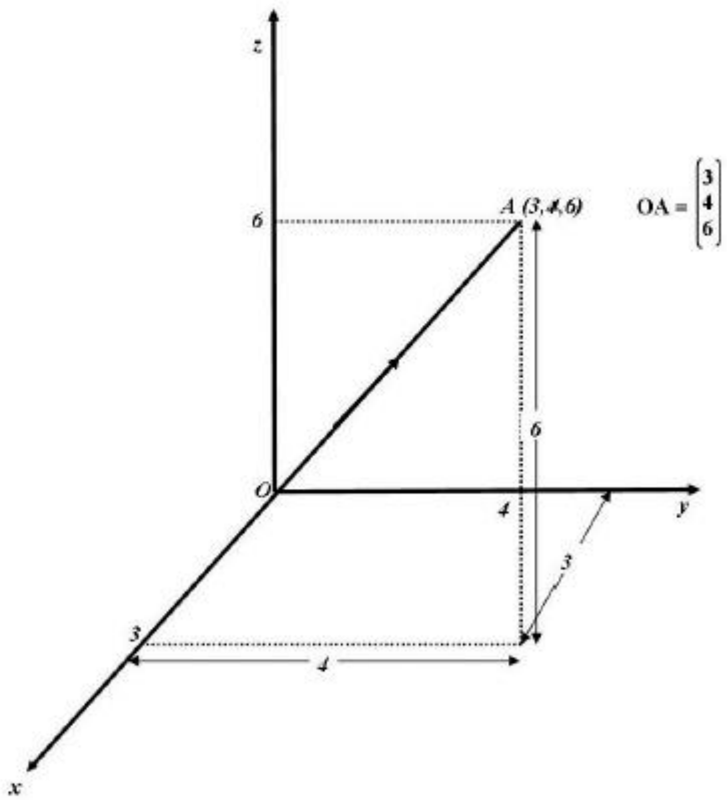
A vector quantity is represented in many ways.

Vectors can be analyzed when represented on a coordinate system.

i) Cartesian or rectangular co-ordinate. *(xy* plane/2 dimension)



ii) *xyz* plane (3 dimension)



Vectors can also be represented in terms of **i, j** and **k.**

i.e **OA** = $\begin{bmatrix}3\\4\end{bmatrix}$; **OA** = $3\mathbf{i} + 4\mathbf{j}$

OA = $\begin{bmatrix}3\\4\\6\end{bmatrix}$; **OA** = $3\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}$

Position vector: It is a vector drawn from the origin of some coordinate system to a point in space to indicate position of object with respect to origin, i.e

OA = $\begin{bmatrix}3\\4\end{bmatrix}$ or **OA** = $\begin{bmatrix}3\\4\\6\end{bmatrix}$

Displacement vector: It is a directed line segment (arrow) whose length indicates the magnitude of the displacement and whose direction is the direction of displacement.

2.4 Operation on vectors

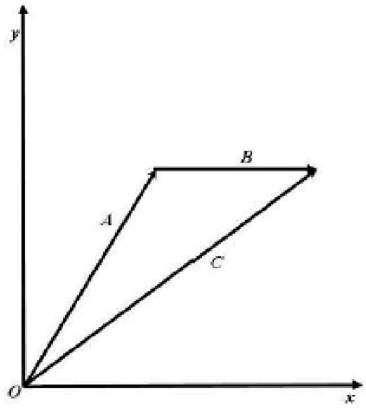
2.4.1 Vector addition

In addition it means two vectors are added to get another vector, i.e

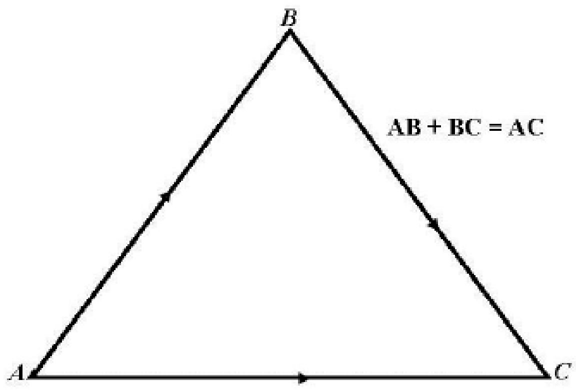
A + B = C

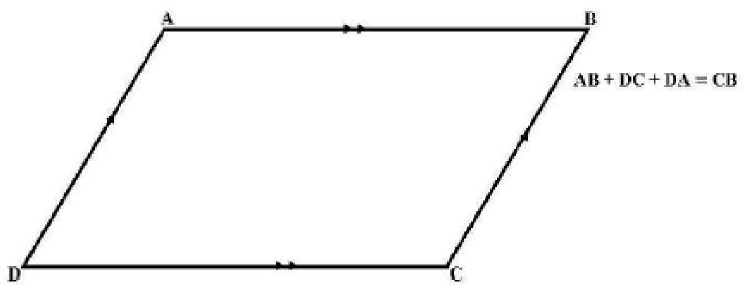
There are two ways of doing this:-

Triangle method: If **A** and **B** are drawn to scale with tail of **B** at the tip of **A**, then **C** is a vector from the tail of **A** to the tip of **B.**



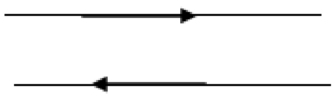
Tip – to – tip Method (polygon): It is an extension of the triangle method to two or more than two vectors.





2.4.2: Vector subtraction

The negative of a vector of equal magnitude but different direction.



A = -**B**

Vector subtraction is vector addition of opposite vectors.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

Example:

1. Given that **A** = 5**i** + 3**j** and **B** = 2**i** - 4**j**

Find: a) **A** + **B** b) **A** - **B**

a) **A** + **B** = 7i - j b) **A** - **B** = 3i - 7j

2.4.3: Multiplication of Vectors

We have two ways of a vector multiplication

- Dot / scalar product
- Cross/ vector product

Dot / Scalar product

Means the result is a scalar.

If there are two vectors **A** and **B** then the dot product of the two vectors is defined as

$\mathbf{A}.\mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos\theta$

Where θ is the angle between the two vectors.

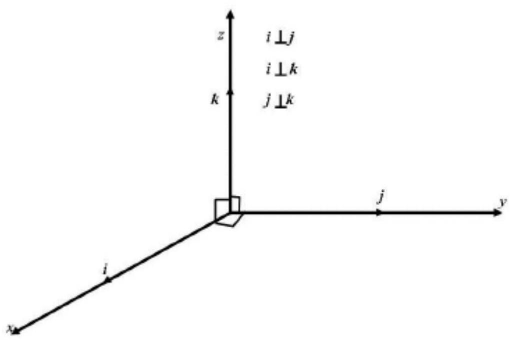
Note: dot product commute, i.e

$\mathbf{A}.\mathbf{B} = \mathbf{B}.\mathbf{A}$

If the vectors are perpendicular to each other then the angle between them is 90^o.

$$\mathbf{A}.\mathbf{B} = |\mathbf{A}||\mathbf{B}| \cos 90^o = 0$$

If we express in terms of **i**, **j** and **k** then



Example

1) Find **A.B** and **B.A** given that

$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k} \text{ and } \mathbf{B} = -\mathbf{j} - 2\mathbf{j} + \mathbf{k}$$

$$\mathbf{A.B} = (2x - 1) + (3 \text{ x } -2) + (4 \text{ x } 1) = {}^-4$$

$$\mathbf{B.A} = ({}^-1 \text{ x } 2) + ({}^-2 \text{ x } 3) + (1 \text{ x } 4) = {}^-4$$

2) Given that, $|\mathbf{A}| = \sqrt{14}$, $|\mathbf{B}| = \sqrt{16}$ and the angle between **A** and **B** is 30°, find **A.B**

$$\mathbf{A.\ B} = |\mathbf{A}||\mathbf{B}| \text{ Cos } \theta = \sqrt{14} \text{ x } \sqrt{16} \text{ Cos } 30^0 = 12.9514$$

Cross Product

Given that two vectors **A** and **B** the cross product of **A** and **B** is defined as

$$\mathbf{A} \text{ x } \mathbf{B} = |\mathbf{A}||\mathbf{B}| \text{ Sin } \theta$$

Consider two vectors

$\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Represent in matrix form $\mathbf{A} \text{ x } \mathbf{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

$$\mathbf{A} \text{ x } \mathbf{B} = \mathbf{i}\,[(a_2 b_3) - (b_2 a_3)] + \mathbf{j}\,[(b_1 a_3) - (a_1 b_3)] + \mathbf{k}\,[(a_1 b_2) - (b_1 a_2)]$$

Example.

Given that $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $\mathbf{B} = -\mathbf{i} + \mathbf{j} + 2\mathbf{k}$

Find **A x B**

$$\mathbf{A} \text{ x } \mathbf{B} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 3 & -1 \\ -1 & 1 & 2 \end{vmatrix}$$

$$\mathbf{A} \text{ x } \mathbf{B} = \mathbf{i}\,[(3 \text{ x } 2) - (1 \text{ x } -1)] + \mathbf{j}\,[(-1 \text{ x } -1) - (2 \text{ x } 2)] + \mathbf{k}\,[(2 \text{ x } 1) - (3 \text{ x } -1)]$$

$$= 7\mathbf{i} - 3\mathbf{j} + 6\mathbf{k}$$

$\mathbf{k.j} = \mathbf{i.k} = 0$, $\mathbf{k.j} = \mathbf{j.k} = 0$ and $\mathbf{i.j} = \mathbf{j.i} = 0$
Also $\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$

Consider vectors $\mathbf{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\text{k}$ and $\mathbf{B} = b_1\text{i} + b_2\mathbf{j} + b_3\mathbf{k}$ then,

$$\begin{aligned} \mathbf{A.B} &= (a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k})\,(b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}) \\ &= a_1b_1\mathbf{i.i} + a_1b_2\mathbf{i.j} + a_1b_3\mathbf{i.k} + a_2b_1\mathbf{j.i} + \\ &\quad a_2b_2\mathbf{j.j} + a_2b_3\mathbf{j.k} + a_3b_1\mathbf{k.i} + a_3b_2\mathbf{k.j} + \\ &\quad a_3b_3\mathbf{k.k} \\ &= a_1b_1 + 0 + 0 + 0 + a_2b_2 + 0 + 0 + 0 + a_3b_3 \\ &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned}$$

2.4.4: Multiplication with scalars

Consider vectors **A, B** and scalar S then

S (**A** + **B**) = S**A** + S**B**

2.4.5: Magnitude and Direction of a vector

Given that **A** = a_1**i** + a_2**j** + a_3**k**, then

|**A**| = $\sqrt{[(a_1)^2 + (a_2)^2 + (a_3)^2]}$

Example:

1. Given that **A** = 2**i** + 3**j** + 4**k**, find |**A**|
   |A| = $\sqrt{[(2)^2 + (3)^2 + (4)^2]}$ = 5.39 units

2. If **A** + **B** + **C** = 0 and **A** = 2**i** + 3**j** + 4**k**, **B** = 5**j** + 6**j** + 7**k**. What is **C**, |**C**| and angle between **C** and x axis.

   C = -**A** –**B** = -7**i** - 9**j** - 11**k**
   |**C**| = $\sqrt{[(-7)^2 + (-9)^2 + (-11)^2]}$ = 15.84 units

   θ= Tan (-9/-7) =52.13°

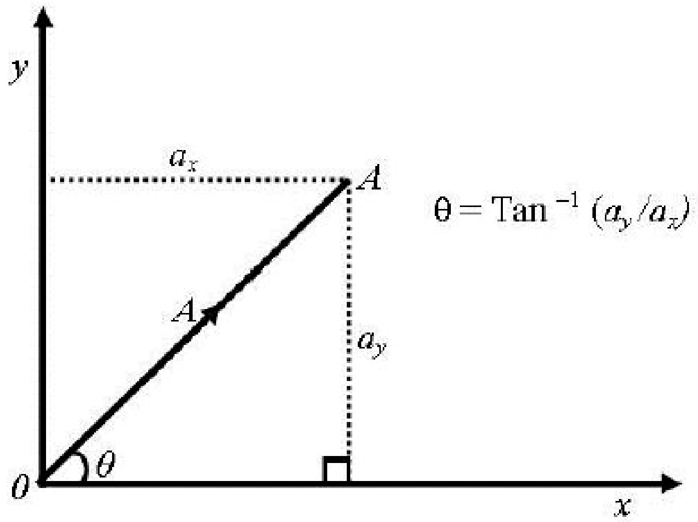
2.4.6: Angle between vectors

We find angles between vectors by using the dot product. This is because dot product gives the result of a scalar.

A.B = |**A**||**B**| Cos θ, θ = $\cos^{-1}\left(\frac{A.B}{|A||B|}\right)$

2.4.7. Angle between vector and axes

Consider vector **A** as shown:-



<u>Examples.</u>

1) Find the magnitude and direction of the following vectors.
   a) $\mathbf{A} = 5i + 3j$ b) $\mathbf{B} = 10i - 7j$ c) $\mathbf{C} = -2i - 3j + 4k$

   <u>Solution</u>

   a) $|\mathbf{A}| = r = \sqrt{5^2 + 3^2} = 5.83$
   $\theta = \text{Cos}^{-1}\left(\frac{5}{5.83}\right) = 30.96^0$

   b) $|\mathbf{B}| = r = \sqrt{10^2 + -7^2} = 12.21$

   $\theta = \text{Cos}^{-1}\left(\frac{10}{12.21}\right) = 35.02^0$

   c) $|\mathbf{C}| = r = \sqrt{-2^2 + -3^2 + 4^2} = 5.39$

   $\theta = \text{Tan}^{-1}\left(\frac{\sqrt{-2^2 + -3^2}}{\sqrt{4}}\right) = 42.03^0$
   $\phi = \text{Tan}^{-1}\left(\frac{-3}{-2}\right) = 56.31^0$

2) The rectangular components of the vectors which lie in $x - y$ plane have their magnitudes and directions given below. Find the x and y components of the vectors.
   a) $r = 10$ and $\theta = 30^0$ b) $r = 7$ and $\theta = 60^0$

   <u>Solution</u>
   a) $x = r \text{ Cos } \theta = 10 \text{ Cos } 30^0 = 8.66$, $y = r \text{ Sin } \theta = 10 \text{ Sin } 30^0 = 5$
   b) $x = r \text{ Cos } \theta = 7 \text{ Cos } 60^0 = 3.5$, $y = r \text{ Sin } \theta = 7 \text{ Sin } 60^0 = 6.06$

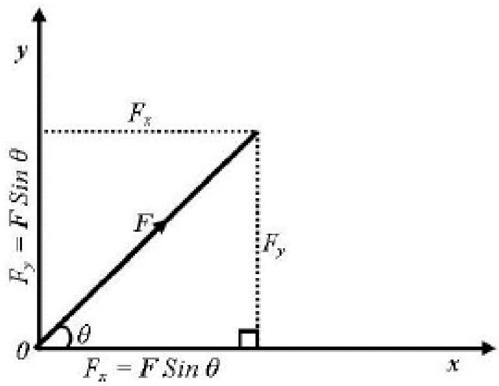
3. a) Find the magnitude and direction of the resultant vector $\mathbf{A} = 5\mathbf{i} + 3\mathbf{j}$ and $\mathbf{B} = 2\mathbf{i} - 4\mathbf{j}$

   <u>Solution</u>
   $\boldsymbol{R} = \boldsymbol{A} + \boldsymbol{B} = 7i - j$
   $|\boldsymbol{R}| = \sqrt{(7^2 + -1^2)} = 7.07$

   $Tan\ \theta = \frac{y}{x} = \frac{-1}{7}$
   $\theta = -8.13^0$

2.4.8 <u>Resolution of Vectors</u>

A component of a vector is the effective part of a vector in that direction.
Consider a Force **F** pulling in the direction as shown.



X component of **F** is **F** Cos θ
Y component of **F** is **F** Sin θ

Example
Consider two forces $\mathbf{F_1}$ and $\mathbf{F_2}$ pulling as shown below. Find the X and Y components of the forces given that

$|\mathbf{F_1}| = 2.88$ and $|\mathbf{F_2}| = 3.44$

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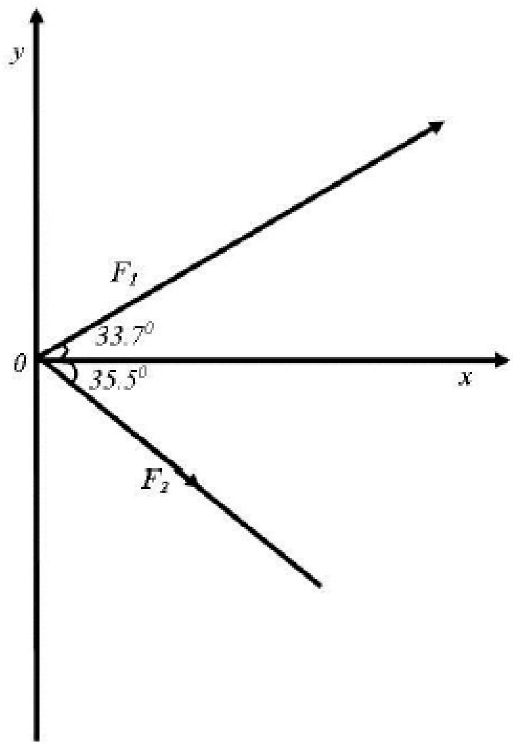

$F_{1x} = |F_1|$ Cos 33.7° = 2.88 Cos 33.7° = 2.40, F_{1y} = $|F_1|$ Sin 33.7° = 2.88 Sin 33.7° = 1.60

$\mathbf{F_1} = 2.4\mathbf{i} + 1.6\mathbf{j}$
Similarly
F_{2x} = $|\mathbf{F_2}|$ Cos 35.5° = 3.44 Cos 35.5° = 2.80, $F_{2y} = |\mathbf{F_2}|$ Sin (-35.5)
= 3.44 Sin (-35.5°) = 2.00

$\mathbf{F_2} = 2.80\mathbf{i} - 2.00\mathbf{j}$

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Sheet No. 2: Vectors

Q1) for the following points:

a) Draw the approximated locations of these points
b) Find the vectors between each pairs of these points
c) Find the angles of the triangle with vertices of these points
d) Find the perimeter and the area of the triangle.

1) A(1,1,3), B(-2,3,1) and C (0,2,-2).
2) A(0,0,3), B(-2,0,1) and C (1,2,0)

Q2) If $V_1=a_1i+b_1j+c_1k$ and $V_2=a_2i+b_2j+c_2k$, Prove the following formulas

1) $V_1 \cdot V_2 = a_1a_2+b_1b_2+c_1c_2$
2) $V_1 \times V_2=0$ if $V_1 // V_2$
3) $V_1 \cdot V_2 =0$ if $V_1 \perp V_2$