Taylor-Maclaurin Series

Consider a function $f(x)$ defined by a power series of the form

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

with radius of convergence $R > 0$. If we write out the expansion of $f(x)$ as

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + \cdots$$

we observe that $f(a) = c_0$. Moreover

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$

$f''(x) = 2c_2 + 3 \cdot 2c_3(x-a) + 4 \times 3c_4(x-a)^2 + \cdots$

$f^{(3)}(x) = 3 \cdot 2c_3 + 4 \cdot 3 \cdot 2c_4(x-a) + \cdots$

$\vdots$

$f^{(n)}(x) = n!\, c_n$

After computing the above derivatives, we observe that

$$f'(a) = c_1, \quad f''(a) = 2c_2 \;\Rightarrow\; c_2 = \frac{f''(a)}{2!} \text{ and } c_3 = \frac{f^{(3)}(a)}{3!}$$

In general, we have

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Suppose that $f(x)$ has a power series expansion at $x = a$ with radius of convergence $R > 0$, then the series expansion of $f(x)$ takes the form:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \cdots$$

Which is called Taylor series.

If $a = 0$,then

$$f(x) = \sum_{n=0}^{\infty} c_n x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \cdots$$

Which is called Maclaurin Series.

Example 1: Compute the Maclaurin series of the following functions

1. $f(x) = e^x$ 2. $f(x) = e^{x^2}$

$f(x) = e^x \quad \Rightarrow \quad f(0) = e^0 = 1$

$f'(x) = e^x \quad \Rightarrow \quad f'(0) = e^0 = 1$

$f''(x) = e^x \quad \Rightarrow \quad f''(0) = e^0 = 1$

$f^{(3)}(x) = e^x \quad \Rightarrow \quad f^{(3)}(0) = e^0 = 1$

1. $e^x = \frac{1}{0!} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

2. $e^{x^2} = \frac{1}{0!} + \frac{x^2}{1!} + \frac{(x^2)^2}{2!} + \frac{(x^2)^3}{3!} + \frac{(x^2)^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}$

Example 2: Compute the Maclaurin series of the following functions

1. $f(x) = \sin x$ 2. $f(x) = \frac{\sin(x^2)}{x^2}$

$f(x) = \sin x \quad \Rightarrow \quad f(0) = \sin 0 = 0$

$f'(x) = \cos x \quad \Rightarrow \quad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \quad \Rightarrow \quad f''(0) = -\sin 0 = 0$

$f^{(3)}(x) = -\cos x \quad \Rightarrow \quad f^{(3)}(0) = -\cos 0 = -1$

We note that $f^{(2n+1)}(x) = (-1)^n \cos x \quad \Rightarrow \quad f^{(2n+1)}(0) = (-1)^n$

$f^{(2n)}(x) = (-1)^n \sin x \quad \Rightarrow \quad f^{(2n)}(0) = 0$

1. $\sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

2. $\sin(x^2) = \frac{(x^2)}{1!} - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n+1}}{(2n+1)!}$

$$\sin(x^2) = \frac{x^2}{1!} - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{(2n+1)!}$$

$$\frac{\sin(x^2)}{x^2} = \frac{1}{1!} - \frac{x^4}{3!} + \frac{x^8}{5!} - \frac{x^{12}}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n+1)!}$$

Taylor Polynomials and Maclaurin Polynomials

The partial sums of Taylor (Maclaurin) series are called Taylor (Maclaurin) polynomials. More precisely, the Taylor polynomial of degree k of $f(x)$ at $x = a$ is the polynomial

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

And the Maclaurin polynomial of degree k of $f(x)$ (at $x = 0$) is the polynomial

$$P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!}x^n$$

Example 3: Compute the Maclaurin polynomial of degree 4 for the function

$$f(x) = \cos x \ln(1-x)$$

Maclaurin polynomial $P_4(x)$ of degree 4 of $f(x)$ is

$$P_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}(0)}{n!}x^n = \frac{f(0)}{0!} + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$f_1(x) = \cos x \quad ⇰ \quad f_1(0) = 1$$

$$f_1^{(2n+1)}(x) = (-1)^n \sin x \quad ⇰ \quad f_1^{(2n+1)}(0) = 0$$

$$f_1^{(2n)}(x) = (-1)^n \cos x \quad ⇰ \quad f_1^{(2n)}(0) = (-1)^n$$

$$f_1(x) = \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

$$f_2(x) = \ln(1-x) \quad ⇰ \quad f_2(0) = \ln(1) = 0$$

$$f_2'(x) = \frac{-1}{1-x} = -(1-x)^{-1} \quad ⇰ \quad f_2'(0) = -1$$

$$f_2''(x) = -(1-x)^{-2} \quad ⇰ \quad f_2''(0) = -1$$

$$f_2^{(3)}(x) = -2(1-x)^{-3} \quad ⇰ \quad f_2^{(3)}(0) = -2$$

$$f_2(x) = \ln(1-x) = 0 - x - \frac{x^2}{2!} - \frac{2x^3}{3!} - \frac{6x^4}{4!} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$$

$$f(x) = \cos x \ln(1-x) = \left(1 - \frac{x^2}{2} + \frac{x^4}{24}\right)\left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}\right)$$

$$P_4(x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^3}{2} + \frac{x^4}{4} = -x - \frac{x^2}{2} + \frac{x^3}{6}$$

Example4 : Compute the first four terms in the power series expansion of following

$$f(x)=\frac{\ln(1+x)}{(1+x)}$$

$$f_1(x)=\ln(1+x) \quad \Rightarrow \quad f_1(0)=\ln(1)=0$$

$$f_1'(x)=\frac{1}{(1+x)}=(1+x)^{-1} \quad \Rightarrow \quad f_1'(0)=1$$

$$f_1''(x)=-(1+x)^{-2} \quad \Rightarrow \quad f_1''(0)=-1$$

$$f_1^{(3)}(x)=2(1+x)^{-3} \quad \Rightarrow \quad f_1^{(3)}(0)=2$$

$$f_1^{(4)}(x)=-6(1+x)^{-4} \quad \Rightarrow \quad f_1^{(4)}(0)=-6$$

$$f_1(x)=\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}$$

$$f_2(x)=\frac{1}{(1+x)}=(1+x)^{-1} \quad \Rightarrow \quad f_2(0)=1$$

$$f_2'(x)=-(1+x)^{-2} \quad \Rightarrow \quad f_2'(0)=-1$$

$$f_2''(x)=2(1+x)^{-2} \quad \Rightarrow \quad f_2''(0)=2$$

$$f_2^{(3)}(x)=-6(1+x)^{-3} \quad \Rightarrow \quad f_2^{(3)}(0)=-6$$

$$f_2(x)=\frac{1}{(1+x)}=1-x+x^2-x^3+x^4$$

$$f(x)=\frac{\ln(1+x)}{(1+x)}=\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}\right)(1-x+x^2-x^3+x^4)$$

$$P_4(x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}-x^2+\frac{x^3}{2}-\frac{x^4}{3}+x^3-\frac{x^4}{2}-x^4$$

$$P_4(x)=x-\frac{3x^2}{2}+\frac{11x^3}{6}-\frac{25x^4}{12}$$

Exercises

1.Compute the Maclaurin series of the following functions

$a.\ \ f(x)=\tan^{-1}x$ $\qquad\qquad$ $b.\ \ f(x)=\sqrt{1+x}$

2.Compute the first four terms in the power series expansion of following

$a.\, f(x)=\sqrt{1+x}\ \cos x$ $\qquad\qquad$ $b.\ \ f(x)=\tan^{-1}x\ \ln(1+x)$