المحاضرة الرابعة

Demoivre's formula

Let $z = r(cos\theta + isin\theta)$ then

$z^n = [r(cos\theta + isin\theta)]^n = r^n(\cos n\theta + i \sin n\theta) = r^n e^{in\theta} \quad , n \in \mathbb{Z}$

Ex. write $(\sqrt{3} + i)^7$ in the form $x + iy$.

Sol. $z = \sqrt{3} + i$

$r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

$$z^7 = 2^7 \left[cos\frac{\pi}{6} + isin\frac{\pi}{6}\right]^7$$

$$= 128\left[cos\frac{7\pi}{6} + isin\frac{7\pi}{6}\right]$$

$$= 128\left[-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right]$$

$$= -64\sqrt{3} - 64i$$

ملاحظة مهمة

1. في الربع الأول كل النسب موجبة
2. في الربع الثاني الجيب والقاطع تمام موجب والبقية سالب
3. في الربع الثالث الظل ومقلوبه موجب والبقية سالب
4. الجيب تمام والقاطع موجب والبقية سالب.

Ex. write $z = \left[\frac{-1+\sqrt{3}i}{\sqrt{3}-i}\right]^5$ in the form $x + yi$

Sol. $z = \left[\frac{-1+\sqrt{3}i}{\sqrt{3}-i}\right].\left[\frac{\sqrt{3}+i}{\sqrt{3}+i}\right] \Longrightarrow \frac{-2\sqrt{3}+2i}{4} = \frac{-\sqrt{3}}{2} + \frac{1}{2}i$

$$r = \sqrt{\left(\frac{-\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \Longrightarrow \sqrt{\frac{3}{4} + \frac{1}{4}} = 1$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \Longrightarrow \theta = \tan^{-1}\left(\frac{\frac{1}{2}}{\frac{-\sqrt{3}}{2}}\right) \Longrightarrow \theta = \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right) = -\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= -\frac{\pi}{6}$$

$$z = r(cos\theta + isin\theta)$$

$$z = 1\left(cos -\frac{\pi}{6} + isin -\frac{\pi}{6}\right)$$

$$z = cos\frac{\pi}{6} - isin\frac{\pi}{6}$$

$$z^5 = cos\frac{5\pi}{6} - isin\frac{5\pi}{6}$$

$$= -\frac{\sqrt{3}}{2} - \frac{1}{2}i$$

Ex. Prove that by Demoivre's formula

1. $\cos 2\theta = \cos\theta^2 - \sin\theta^2$

2. $\sin 2\theta = 2\sin\theta\cos\theta$

Proof.

$$(\cos\theta + isin\theta)^2 = \cos\theta^2 + 2isin\theta cos\theta + i^2 sin\theta^2$$

$$cos2\theta + isin2\theta = \cos\theta^2 - sin\theta^2 + 2isin\theta cos\theta$$

$$cos2\theta = \cos\theta^2 - sin\theta^2 - - - - - - - - - - 1$$

$$sin2\theta = 2sin\theta cos\theta - - - - - - - - - - - - 2$$

Ex. Prove that by Demoivre's formula

1. $cos3\theta = 4cos\theta^3 - 3cos\theta$

2. $\sin 3\theta = 3sin\theta - 4sin^3\theta$

Proof. $(cos\theta + isin\theta)^3 = cos\theta^3 + 3cos\theta^2.isin\theta + 3cos\theta.i^2sin\theta^2 + i^3sin\theta^3$

$$cos3\theta + isin3\theta = cos\theta^3 + 3icos\theta^2sin\theta - 3cos\theta sin\theta^2 - isin\theta^3$$

$$cos3\theta = cos\theta^3 - 3cos\theta sin\theta^2 = cos\theta^3 - 3(1 - cos\theta^2).cos\theta$$

$$= cos\theta^3 - 3cos\theta + 3cos\theta^3 = 4cos\theta^3 - 3cos\theta$$

$$cos3\theta = 4cos\theta^3 - 3cos\theta - - - - - - - - 1$$

$$sin3\theta = 3cos\theta^2sin\theta - sin\theta^3$$

$$= 3(1 - sin\theta^2)sin\theta - sin\theta^3$$

$$= 3sin\theta - 3sin\theta^3 - sin\theta^3$$

$$sin3\theta = 3sin\theta - 4sin\theta^3 - - - - - - -2$$

حل واجبات سابقة (مناقشة)

H .W. Show that

(a) $Re\,(iz) = -\,Im(z)$.

$(b)\ Im(iz)\ =\ Re(z)$.

(a) Sol.

Let $z = x + iy \Rightarrow Re(ix - y) = -y$ ------------ (1)

$-\,Im(x + iy) \Rightarrow -y$ -------------- (2)

$\therefore \quad (1) = (2)$

$(\boldsymbol{b})$ **Sol.**

Let $z = x + iy \Longrightarrow Im(ix - y) = x$ -------------- (1)

$Re(z) = Re(x + iy) = x$ ---------- (2)

$\therefore \quad (1) = (2)$

1- Find the value of x, y that satisfy the eq.

$$(x - y - 6) + i\,(y^2 - x) = 0$$

Sol.

$(x - y - 6) = 0$ -----------1

$y^2 - x = 0$ ---------------2

From -1- $x = y + 6$ --------- 3 Substitute -2-

$y^2 - y - 6 = 0$

$(y - 3)(y + 2) \Longrightarrow y = 3, -2$ and $x = 8,9$

2. Solve for real x, y the eq.

$$\left(\frac{1 + i}{1 - i}\right)^2 + \frac{1}{x + iy} = 1 + i$$

Sol.

$$\left(\frac{1 + i}{1 - i} \times \frac{1 + i}{1 + i}\right)^2 + \frac{1}{x + iy} = 1 + i \qquad \Longrightarrow \left(\frac{2i}{2}\right)^2 + \frac{1}{x + iy} = 1 + i$$

$$-1 + \frac{1}{x + iy} = 1 + i \qquad \Longrightarrow \frac{1}{x + iy} = 2 + i$$

$$x + iy = \frac{1}{2+i} \times \frac{2-i}{2-i}$$

$$x + iy = \frac{2}{5} - \frac{i}{5}$$

$$\therefore x = \frac{2}{5} \qquad y = \frac{i}{5}$$

3. Solve the following equation $(\mathbf{3} - \mathbf{2}\boldsymbol{i})\ (\boldsymbol{x} + \boldsymbol{iy}) = \mathbf{2}(\boldsymbol{x} - \mathbf{2}\boldsymbol{iy}) + \mathbf{2}\boldsymbol{i} - \mathbf{1}$

Sol.

$$3x + 3yi - 2xi + 2y = 2x - 4iy + 2i - 1$$

$3x + 2y = 2x - 1 \Rightarrow x + 2y = -1$--------- .1.

$3y - 2x = 2 - 4y \Rightarrow 7y - 2x = 2$ ------- .2.

$x + 2y = -1$--------- .1

$7y - 2x = 2$ ------- .2.

$$x = -1, y = 0$$

5. Express the following equations in complex conjugate form.

عبر عن المعادلات الآتية بصبغة المرافق العقدي.

1) $2\text{x} + \text{y} = 5$, 2) $x^2 + \text{y}^2 = 36$

1) **Sol.**

$$2\left(\frac{z + \bar{z}}{2}\right) + \left(\frac{z - \bar{z}}{2i}\right) = 5 \quad \Rightarrow z + \bar{z} + \left(\frac{z - \bar{z}}{2i}\right) = 5 \quad \times 2$$

$$2z + 2\bar{z} + \left(\frac{z - \bar{z}}{i} \times \frac{i}{i}\right) = 10$$

$$2z + 2\bar{z} - iz + i\bar{z} = 10 \quad or \qquad 2z + 2\bar{z} + (\bar{z} - z)i = 10$$

2) Sol.

$z\bar{z} = 36$

اثبت أن :معادلة القطع الزائد

$$x^2 - y^2 = 1$$

هي

$$z^2 + (\bar{z})^2 = 2$$

Sol.

$x^2 - y^2 = 1$

$$\left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 = 1$$

$$\frac{z^2}{4} + \frac{2\,z\bar{z}}{4} + \frac{\bar{z}^2}{4} - \left(\frac{z^2}{4i^2} - \frac{2\,z\bar{z}}{4i^2} + \frac{\bar{z}^2}{4i^2}\right) = 1$$

$$\frac{z^2}{4} + \frac{z\bar{z}}{2} + \frac{\bar{z}^2}{4} + \frac{z^2}{4} - \frac{z\bar{z}}{2} + \frac{\bar{z}^2}{4} = 1$$

$$\frac{2z^2}{4} + \frac{2\bar{z}^2}{4} = 1 \Longrightarrow \left[\frac{z^2}{2} + \frac{\bar{z}^2}{2} = 1\right] \times 2$$

$$z^2 + (\bar{z})^2 = 2$$

H.w. 1.Express z in the form $x + yi$ when $z = -5e^{\frac{5\pi i}{6}}$

Sol.

$z = r[\cos\theta + isin\theta]$

$$z = -5\left[\cos\left(\frac{5\pi}{6}\right) + isin\left(\frac{5\pi}{6}\right)\right]$$

$$z = -5\left[-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right]$$

$$z = \frac{5\sqrt{3}}{2} - \frac{5i}{2}$$

2. Show that $\left|e^{i\theta}\right| = 1$

Sol.

$\left|e^{i\theta}\right| = |cos\theta + isin\theta| \Longrightarrow \sqrt{(cos\theta)^2 + (sin\theta)^2} = 1$

3. Find $\left|e^{\frac{\pi}{2}i}\right|$

Sol.

$\left|cos\frac{\pi}{2} + isin\frac{\pi}{2}\right| \Longrightarrow \sqrt{\left(cos\frac{\pi}{2}\right)^2 + \left(sin\frac{\pi}{2}\right)^2} = 1$