Method (6)

Tabular Integration

Consider the integral of the form $\int f(x)\,g(x)dx$ in which $\int f(x)$ can be differential repeatedly to Zero and $g(x)$ can be integral repeatedly without difficulty Tabular integration save a great deal of work as natural method consider from integration

$f(x)$ and Its derivative		$g(x)$ and Its Integrals
$f(x)$	+	$g(x)$
$f'(x)$	-	$\int g(x)dx = g_1(x)$
$f''(x)$	+	$\int g_1(x)dx = g_2(x)$
$f'''(x)$		$\int g_2(x)dx = g_3(x)$
$\vdots$		$\vdots$
$f^{n-1}(x)$		$g_{n-1}(x)dx = g_n(x)$
$f^{n}(x) = 0$		

$$I = f(x)g_1(x) - f'(x)g_2(x) + f''(x)g_3(x) - \ldots\ldots\ldots \pm f^{n-1}(x)g_n(x)$$

Examples:

1) **Evaluate**

$$I = \int x^2 e^x dx$$

Solution :-

$f(x)$ and Its derivative		$g(x)$ and Its Integrals
x^2	+	e^x
$2x$	-	$\int e^x dx = e^x$
2	+	$\int e^x dx = e^x$
0		$\int e^x dx = e^x$

$$I = \int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + c$$

2) Evaluate

$$I = \int \left(x^3 - 2x^2 + 3x + 1\right)\sin(2x)dx$$

Solution :-

$f(x)$ and Its derivative		$g(x)$ and Its Integrals
$x^3 - 2x^2 + 3x + 1$	+	$\sin(2x)$
$3x^2 - 4x + 3$	-	$\int \sin(2x)dx = -\frac{1}{2}\cos(2x)$
$6x - 4$	+	$\int -\frac{1}{2}\cos(2x)dx = -\frac{1}{4}\sin(2x)$
6	-	$\int -\frac{1}{4}\sin(2x)dx = \frac{1}{8}\cos(2x)$
0		$\int \frac{1}{8}\cos(2x) = \frac{1}{16}\sin(2x)$

$$I = \int (x^3 - 2x^2 + 3x + 1)\sin(2x)\,dx$$

$$= (x^3 - 2x^2 + 3x + 1)\frac{-1}{2}\cos\ (2x) - (3x^2 - 4x + 3)\frac{-1}{4}\sin\ (2x) + 6x$$

$$- 4)\frac{1}{8}\cos\ (2x) - 6\frac{1}{16}\sin\ (2x)$$

3) Evaluate

$$I = \int e^x \sin(x)dx$$

Solution :- $u = e^x \qquad dv = \sin(x)dx$

$du = e^x dx \qquad v = -\cos(x)$

$$I = -e^x\cos(x) + \int e^x\cos(x)dx = -e^x\cos(x) + j$$

Where $u = e^x \quad dv = \cos(x)dx$

$du = e^x dx \quad v = \sin(x)$

$$j = \int e^x\cos(x)dx \Rightarrow$$

$$j = e^x\sin(x) - \int e^x\sin(x)dx = e^x\sin(x) - I$$

$$I = -e^x\cos(x) + e^x\sin(x) - I \;\Rightarrow\; 2I = -e^x\cos(x) + e^x\sin(x) \Rightarrow = \frac{1}{2}e^x\left(\cos(x) - \sin(x)\right)$$

Method (7)

Integration Of Rational Functions

Definition :- A rational function is a quotient of two polynomials as $R(x) = \frac{P_n(x)}{Q_m(x)}$ $Q_m \neq 0$ where $P_n(x)$ and $Q_m(x)$ are polynomial of degree n and m.

Note: If $n > m$ we perform a long division until we obtain a rational function whose number numerator degree than or equal to the denominator degree.

Example: Evaluate $I = \int \frac{3x^3 - x^2 - 5x + 1}{x-2} \; dx$

Solution:

$$3x^2 + 5x + 5$$

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$$
\begin{array}{r|l}
 & x^2-6x-2 \\ \hline
x-2 & 3x^3-x^2-5x+1 \\
 & \mp 3x^3 \pm 6x^2 \\ \hline
 & 5x^2-5x+1 \\
 & \mp 5x^2 \pm 10x \\ \hline
 & 5x+1 \\
 & \mp 5x \pm 10 \\ \hline
 & 11
\end{array}
$$

$$I = \int 3x^2 + 5x + 5 + \frac{11}{x-2} = x^3 + \frac{5}{2}x^2 + 5x + 11\ln|x-2| + c$$

Note: If $n \le m$ we have to discuss three cases of separating $\frac{P_n(x)}{Q_m(x)}$ as a sum partial fractions.

Case (1) **If the m factors of $Q_m(x)$ are all different and simple , that is $Q_m(x) = (x-a_1)(x-a_2)\ldots\ldots(x-a_{m)}$, then we assign the sum of m partial fractions to these factors as follows :-**

$$\frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \ldots\ldots + \frac{A_m}{(x-a_m)} \quad \textbf{where } A_1, A_2, \ldots, A_m \textbf{ are constant}$$

Example: Evaluate

$$I = \int \frac{x^2+3x+3}{x^3-x}dx = \int \frac{x^2+3x+3}{x(x-1)(x+1)}dx$$

Solution :- $\dfrac{x^2+3x+3}{x(x-1)(x+1)}=\dfrac{A}{x}+\dfrac{B}{(x-1)}=\dfrac{C}{(x+1)}=\dfrac{A(x-1)(x+1)+Bx(x+1)+cx(x-1)}{x(x-1)(x+1)}$

$x^2+3x+3=A(x-1)(x+1)+Bx(x+1)+cx(x-1)$

$at \quad x=0 \Rightarrow 3=A(0-1)(0+1)+0+0 \quad \Rightarrow A=-3$

$at \quad x=1 \Rightarrow 7=0+B(1)(1+1)+0 \quad \Rightarrow B=7/2$

$at \quad x=-1 \Rightarrow 1=0+0+C(-1)(-1-1) \quad \Rightarrow C=1/2$

Or

$x^2+3x+3=Ax^2-A+Bx^2+Bx+Cx^2-Cx$

$=(A+B+C)x^2+(B-C)x-A$

$$\Rightarrow \left| \begin{array}{l} A+B+C=1 \\ B-C=3 \\ -A=3 \end{array} \right. \quad \Rightarrow A=-3 \quad B=7/2 \quad C=1/2$$

$$I=\int\left(\frac{-3}{x}+\frac{7/2}{x-1}+\frac{1/2}{x+1}\right)dx=-3\ln(x)+\frac{7}{2}\ln(x-1)+\frac{1}{2}\ln(x+1)+$$

Case (2) **Repeated factors of** $Q_m(x)$

Suppose $(x-a)^r$ **is the highest power of** $(x-a)$ **which divided** $Q_m(x)$ **then to this factor we assign the sum of** r **partial fractional as follows:-**

$$\frac{A_1}{(x-a)}+\frac{A_2}{(x-a)^2}+\ldots\ldots+\frac{A_r}{(x-a)^r}$$ **where** $A_1, A_2, \ldots, A_r$ **are constant**

Evaluate $I=\displaystyle\int\frac{x^3-3x^2+4x-2}{x(x-1)^2(x+1)(x+2)^3}dx$

Solution :- $\frac{x^3-3x^2+4x-2}{x(x-1)^2(x+1)(x+2)^3}=\frac{A}{x}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}+\frac{D}{X+1}+\frac{E}{(x+2)}+\frac{F}{(x+2)^2}+\frac{G}{(x+2)^3}$

Evaluate $I=\int \frac{(x+5)}{(x+2)(x-1)^2}dx$

Solution :- $\frac{(x+5)}{(x+2)(x-1)^2}=\frac{A}{(x+2)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2}=\frac{A(x-1)^2+B(x+2)(x-1)+C(x+2)}{(x+2)(x-1)^2}$

Case (3) **Repeated factors of** $Q_m(x)$ **is** (x^2+ax+b) **is not analysis we let** $(ax+b)$

For Example :- $\frac{1}{x^2+1}=\frac{ax+b}{x^2+1}$ **because** (x^2+1) **is not analysis**

Evaluate $I=\int \frac{x}{(x^2+1)(x+1)^2}dx$

Solution :- $\frac{x}{(x^2+1)(x+1)^2}=\frac{Ax+B}{(x^2+1)}+\frac{C}{(x+1)}+\frac{D}{(x+1)^2}$

Method (8)

Integration Of Irrational Functions

If the integral contain a single irrational expression the from

$\sqrt[q]{(ax+b)}=(ax+b)^{\frac{1}{q}}$ **Let** $z=(ax+b)^{\frac{1}{q}}\Rightarrow z^q=ax+b\Rightarrow qz^{q-1}=adx\Rightarrow dx=\frac{q}{a}z^{q-1}dz$

Evaluate $I=\int \frac{2x+3}{\sqrt{x+2}}dx=\int \frac{2x+3}{(x+2)^{1/2}}dx$

Solution :- **Let** $z=(x+2)^{1/2} \Rightarrow z^2=x+2 \Rightarrow 2zdz=dx$

$$\Rightarrow I=\int \frac{2(z^2-2)+3}{z}\,2zdx=2\int(2z^2-1)dz=2\left(\frac{2}{3}z^3-z\right)+c=2\left(\frac{2}{3}(x+2)^{3/2}-(x+2)^{1/2}\right)+c$$

Evaluate $I=\int \frac{dx}{\sqrt[3]{x^2}+\sqrt{x}}=\int \frac{dx}{x^{2/3}+x^{1/2}}$

Solution :- **Let** $z=(x)^{1/6} \Rightarrow z^6=x \Rightarrow 6z^5dz=dx$

$$I=\int \frac{6z^5\,dz}{z^4+z^3}=6\int \frac{z^5\,dz}{z^3(z+1)}=6\int \frac{z^2\,dz}{z+1}=6\int (z-1+\frac{1}{z+1})dz$$

$$=6\left(\frac{1}{2}z^2-z+\ln(z+1)\right)+c=6\left(\frac{1}{2}x^{1/3}-x^{1/6}+\ln(x^{1/6}\right)+c$$

Evaluate $I=\int \frac{\sqrt{x}}{1+\sqrt[4]{x}}dx=\int \frac{x^{1/2}}{1+x^{1/4}}dx$

Solution :- **Let** $z=(x)^{1/4} \Rightarrow z^4=x \Rightarrow 4z^3dz=dx$

$$I=\int \frac{z^2.4z^3}{1+z}dz=4\int \frac{z^5}{z+1}dz=4\int (z^4-z^3+z^2-z+1-\frac{1}{z+1})dz$$

$$=\left(\frac{1}{5}z^5-\frac{1}{4}z^4+\frac{1}{3}z^3-\frac{1}{2}z^2+z-\ln(z+1)+c\right)$$

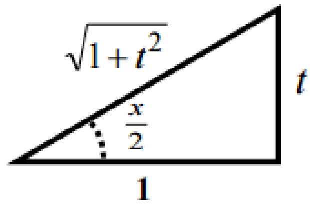
Method (9)

Integration Of Rational Functions of Trigonometric of sine and cosine

If the integral is a rational function of trigonometric substitution of $t=\tan(\frac{x}{2})$ **Will reduce the integral to a relational function of** t **which can be handle by method 7 mathematically speaking**

$$t=\tan(\frac{x}{2}) \Rightarrow \frac{x}{2}=\tan^{-1}(t) \Rightarrow \frac{dx}{2}=\frac{dt}{1+t^2} \Rightarrow \boxed{dx=\frac{2dt}{1+t^2}}$$

$$\sin(\frac{x}{2})=\frac{t}{\sqrt{1+t^2}} \quad , \quad \cos(\frac{x}{2})=\frac{1}{\sqrt{1+t^2}}$$



$$\sin(x)=2\sin(\frac{x}{2})\cos(\frac{x}{2})=\frac{2t}{1+t^2} \Rightarrow \boxed{\sin(x)=\frac{2t}{1+t^2}}$$

$$\cos(x)=\cos^2(\frac{x}{2})-\sin^2(\frac{x}{2})=\frac{1-t^2}{1+t^2} \Rightarrow \boxed{\cos(x)=\frac{1-t^2}{1+t^2}}$$

Evaluate $I=\int \frac{dx}{4-4\cos(x)}dx$

Solution :- $I=\int \frac{dx}{4-4\cos(x)}dx=\int \frac{\frac{2}{1+t^2}dt}{5-4\left(\frac{1-t^2}{1+t^2}\right)}=\int \frac{2dt}{5(1+t^2)-4(1-t^2)}$

$$=2\int \frac{dt}{1+9t^2}=\frac{2}{3}\int \frac{3dt}{1+(3t)^2}=\frac{2}{3}\tan^{-1}(3t)+c=\frac{2}{3}\tan^{-1}[3\tan(\frac{x}{2})]+c$$

Evaluate $I=\int \frac{dx}{3\cos(x)+4\sin(x)}$

Solution:-

$$I=\int \frac{\frac{2dt}{1+t^2}}{3\left(\frac{1-t^2}{1+t^2}\right)+4\left(\frac{2t}{1+t^2}\right)}=2\int \frac{dt}{3(1-t^2)+8t}=2\int \frac{dt}{3-3t^2+8t}=-2\int \frac{dt}{3t^2-8t-3}$$

$$I=-2\int \frac{dt}{3t^2-8t-3}=\int \frac{dt}{(3t+1)(t-3)}$$

$$\frac{1}{(3t+1)(t-3)}=\frac{A}{(3t+1)}+\frac{B}{(t-3)} \Rightarrow \frac{1}{(3t+1)(t-3)}=\frac{A(t-3)+B(3t+1)}{(3t+1)(t-3)}$$

$1=A(t-3)+B(3t+1) \qquad A=-\frac{3}{10} \qquad B=\frac{1}{10}$

$$I=-2\int\left(\frac{-3/10}{3t+1}+\frac{1/10}{t-3}\right)dt=-2\left(-\frac{1}{10}\int \frac{3dt}{3t+1}+\frac{1}{10}\int \frac{dt}{t-3}\right)=\frac{1}{5}\ln(3t+1)-\frac{1}{5}\ln(t-3)+c$$

Good Luck