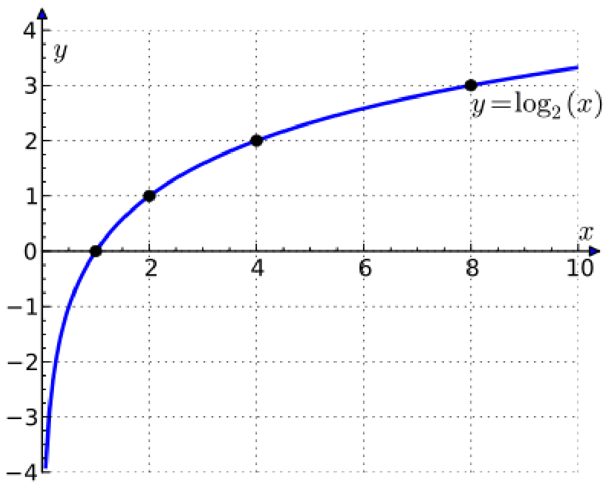
Exponential and Logarithm Functions

Logarithm Functions

The logarithm function with base b is the function $y = \log_b x$ where $b > 0, b \neq 1$ for all $x > 0$.

Here is the graph of $y = \log_2 x$.



Notes:

1. For any base, the x-intercept is 1. $\equiv$ $\log_b 1 = 0$.
2. The graph passes through the point $(b, 1)$. $\equiv$ $\log_b b = 1$.
3. The graph is below the x-axis for $0 < x < 1$ $\equiv$ $\log_b x < 0 \;\forall\; x \epsilon (0,1)$
4. The function is defined only for positive values of x.
5. The range of the function is all real numbers.
6. The negative y-axis is a vertical asymptote.
7. $\log_b(xy) = \log_b x + \log_b y$.
8. $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$.
9. $\log_b\left(\frac{1}{x}\right) = -\log_b x$.
10. $\log_b x^y = y \log_b x$.
11. For each strictly positive real number a and b, different from 1, we have

$$\log_b x = \frac{\log_a x}{\log_a b}$$

The Natural Logarithm $y = \ln x$

The system of natural logarithms has the number called e as its base. e is an irrational number; its decimal value is approximately 2.71828182845904.
To indicate the natural logarithm of a number we write "ln." $\ln x$ means $\log_e x$.
So, we have

1. $\ln e = 1$
2. $\log_b x = \dfrac{\ln x}{\ln b}$
3. $\ln(xy) = \ln x + \ln y$
4. $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$
5. $\ln x^n = n \ln x$

Derivative of Natural Logarithm Function

If u is a function x, then

$$\frac{d}{dx}(\ln u) = \frac{1}{u} \cdot \frac{du}{dx}$$

Example 1: Find derivatives of the functions

1. $y = \ln(5x + 1)$

$$\frac{dy}{dx} = \frac{1}{5x+1} \times 5 = \frac{5}{5x+1}$$

2. $y = 2x \tan^{-1} x - \ln(x^2 + 1)$

$$\frac{dy}{dx} = \frac{2x}{1+x^2} + 2\tan^{-1} x - \frac{2x}{x^2+1} = 2\tan^{-1} x$$

3. $y = \ln(\sin 3x)$

$$\frac{dy}{dx} = \frac{1}{\sin 3x} \times 3\cos 3x = 3\cot 3x$$

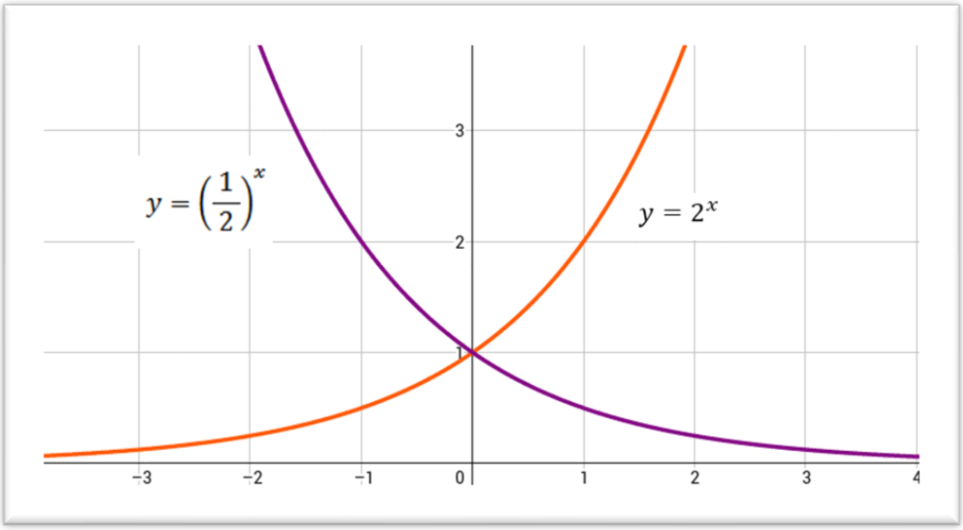
4. $y = \ln(x^2 + 3)^{(x^2+3)}$ ⇨ $y = (x^2 + 3)\ln(x^2 + 3)$

$$\frac{dy}{dx} = (x^2 + 3) \times \frac{2x}{(x^2+3)} + 2x\ln(x^2 + 3) = 2x + 2x\ln(x^2 + 3)$$

Exponential Functions

For any positive number $a > 0$, $a \neq 1$, there is a function called an exponential function that is defined as $f(x) = a^x$

Here is the graph of $y = 2^x$ and $y = \left(\frac{1}{2}\right)^x$



Now, let's talk about some of the properties of exponential functions.

1. The graph of $f(x) = a^x$ will always contain the point$(0,1)$. Or put another way, $a^0 = 1$ for any a.
2. For every possible a, $a^x > 0$. Note that this implies that $a^x \neq 0$.
3. If $0 < a < 1$ then the graph of a^x will decrease as we move from left to right.
4. If $a > 1$ then the graph of a^x will increase as we move from left to right.
5. If $a^x = b^x$ then $a = b$.

Basic rules for exponents

1. The product rule $a^x . a^y = a^{x+y}$
2. The quotient rule $\dfrac{a^x}{a^y} = a^{x-y}$
3. The rule for power of a power $(a^x)^y = a^{x.y}$

Natural Exponential Function

The function $f(x) = e^x$ is often called exponential function or natural exponential function which is an important function. The exponential function $f(x) = e^x$ is the inverse of the logarithm function $f(x) = \ln x$.

Derivatives of Exponential Function

If u is a function x, then

$$\frac{d}{dx}(e^u) = e^u . \frac{du}{dx}$$

Example 2: Find y' of the functions at $x = 0$

1. $y = e^{3x-2}$

   $y' = e^{3x-2} \times 3 = 3e^{3x-2}$

   $y'(0) = 3e^{3(0)-2} = 3e^{-2} = 3 \times 0.135 = 0.405$

2. $y = 2xe^{1-5x}$

   $y' = 2xe^{1-5x} \times (-5) + 2e^{1-5x} = e^{1-5x}(-10x + 2)$

   $y'(0) = e^{1-5(0)}(-10(0) + 2) = 2e$

3. $y = e^{-3x} \sin 2x$

   $y' = 2e^{-3x} \cos 2x - 3e^{-3x} \sin 2x = e^{-3x}(2\cos 2x - 3 \sin 2x)$

   $y'(0) = e^0(2\cos 0 - 3 \sin 0) = 2$

4. $y = e^{\sqrt{1-2x}}$

   $$y' = \frac{-2e^{\sqrt{1-2x}}}{2\sqrt{1-2x}} = \frac{-e^{\sqrt{1-2x}}}{\sqrt{1-2x}}$$

   $$y'(0) = \frac{-e^{\sqrt{1-2(0)}}}{\sqrt{1-2(0)}} = -e$$

Solving Exponential and Logarithm Equations

Logarithms are the "opposite" of exponentials. In practical terms, I have found it useful to think of logarithm in terms of the relationship:

$$y = \log_b x \quad \Leftrightarrow \quad x = b^y$$

$$y = \ln x \quad \Leftrightarrow \quad x = e^y$$

Example 3: Find the value of x

1. $2^x = 64 \quad \Rightarrow \quad 2^x = 2^6$

   $\Rightarrow \quad x = 6$

2. $10^x = 25 \quad \Rightarrow \quad x = \log_{10} 25 \quad \Rightarrow \quad x = 1.398$

3. $5e^x = 20 \quad \Rightarrow \quad e^x = 4$

   $\Rightarrow \quad \ln e^x = \ln 4$

   $\Rightarrow \quad x = 1.386$

4. $\ln(2x+1) = 3.045 \quad \Rightarrow \quad e^{\ln(2x+1)} = e^{3.045}$

   $\Rightarrow \quad 2x + 1 \cong 21$

   $\Rightarrow \quad x \cong 10$

5. $5^{2x} = 35 \quad \Rightarrow \quad \ln 5^{2x} = \ln 35$

   $\Rightarrow \quad 2x \ln 5 = 3.555$

   $\Rightarrow \quad 2x \times 1.61 = 3.555$

   $\Rightarrow \quad 3.22x = 3.555$

   $\Rightarrow \quad x = \dfrac{3.555}{3.22} = 1.1$

Example 4: pH is a measure of acidity and is given by the formula $\text{pH} = -\log[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration (moles /liter of solution). Determine the hydrogen ion concentration if the pH of a solution is 4.

$4 = -\log[\text{H}^+] \quad \Rightarrow \quad [\text{H}^+] = 10^{-4}$ moles /liter

Example 5: The loudness L of a sound in decibels (dB), is given by the formula

$L = 10\log(I/I_0)$ where I represents the intensity of the sound in w/m^2.

Determine the intensity of an alarm that emits 120 dB of sound if

$I_0 = 10^{-12}\ w/m^2$.

$$120 = 10\log(I/10^{-12}) \quad ⇨ \quad 12 = \log(I/10^{-12})$$

$$\frac{\mathrm{I}}{10^{-12}} = 10^{12} \quad ⇨ \quad I = 1\ \ w/m^2$$

Radioactive Decay

The model $M(t) = M_0 e^{-\frac{\ln 2}{h}t}$ represents the amount of material (mass) left after a given amount of time t, where M_0 is the initial mass size and h is the half-life of the material.

Example 6: The half-life of radioactive radium is 1600 years. If a sample initially contains 50 gm, how long will it be until it contains 45 gm?

$$M(t) = M_0 e^{-\frac{\ln 2}{h}t} \quad ⇨ \quad 45 = 50e^{-\frac{\ln 2}{1600}t}$$

$$e^{-\frac{\ln 2}{1600}t} = \frac{45}{50} \quad ⇨ \quad -\frac{\ln 2}{1600}t = \ln 0.9 \quad ⇨ \quad t = -\frac{1600 \times \ln 0.9}{\ln 2}$$

$$t = -\frac{1600\times(-0.105)}{0.693} = 242 \text{ years}$$

Newton's Low of Cooling

The temperature of an object T, in surrounding air with temperature T_s will behave according to the formula $T = D_0 e^{-kt} + T_s$, where D_0 is the difference between the initial temperature of the object and the surroundings and k is a constant, the continuous rate of cooling of the object.

Example 7: Water at a temperature of $80°C$ is placed in a room which is held at a constant temperature of $25°C$. How much would be the temperature of water after 10 minutes if $k = 0.056$.

$$D_0 = 80 - 25 = 55$$

$$T = 55e^{-0.056\times 10} + 25 = 31 + 25 = 56°C$$

Exercises

Find derivative in each of the following problems$(1-4)$

1. $y = \ln(x^2 + x)$
2. $y = x^3 \ln(x^2 - 2x + 5)$
3. $y = e^{\sin^{-1} x}$
4. $y = x^3 e^{-5x}$

Find the value of x in each of the following problems$(5-7)$

5. $10^{2x} = 82$
6. $2e^{3x} = 180$
7. $3^{3x-2} = 82$
8. Determine the intensity of an alarm that emits 30 dB of sound.
9. Determine the hydronium ion concentration in a solution that has a pH of 8.34.

   If $\mathrm{pH} = -\log[H_3O^+]$
10. The half-life of radioactive einsteinium is 276 days. After 100 days, 0.5 gram remains. What was the initial amount?
11.