Solution of Laplace's Equation

In this lecture we discuss the solution of Laplace's equation by using the method of separation of variables. The Laplace equation is written as: $u_{xx} + u_{yy} = 0$.

The following examples explain how to find the solution.

Example 1: Solve the Laplace's equation by the method of separation of variables over $0 \le x \le \pi , y > 0$ with $u(x,0) = u_x(0,y) = u_x(\pi,y) = 0$

Solution: Assume the solution is $u(x,y) = F(x\,)G(y)$ *then* $F''G + FG'' = 0$

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda^2$$

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$$F'' + \lambda^2 F = 0$$

$$F = A\,sin\,\lambda x + B\,cos\,\lambda x$$

$$u_x(0,y) = u_x(\pi,y) = 0 \quad ⇨ \quad F'(0) = 0 \; and \;\; F'(\pi) = 0$$

$$F' = \lambda A\,cos\,\lambda x - \lambda B\,sin\,\lambda x$$

$$F'(0) = 0 \;⇨\; 0 = \lambda A\,cos(0) - \lambda B\,sin(0) \quad ⇨\; A = 0$$

So $F = B\cos\lambda x$

$$F'(\pi) = 0 \quad ⇨\; 0 = -\lambda B\sin(\lambda\pi)$$

Either $B = 0$ ⇨ $u = 0$ (trivial solution)

Or $\sin(\lambda\pi) = 0$ ⇨ $\lambda\pi = n\pi$ ⇨ $\lambda = n$, $\quad n = 1,2,3,\cdots$

So $F = B\cos nx$

Nwo $-\dfrac{G''}{G} = -\lambda^2$ ⇨ $G'' - n^2 G = 0$

$$G = C\sinh ny + D\cosh ny$$

$$u(x,0) = 0 \;⇨ G(0) = 0 \;⇨\; 0 = C\sinh(0) + D\cosh(0) \;⇨\; D = 0$$

$$G = C\sinh ny$$

Then $u(x,y) = B\cos nx \times C\sinh ny$

$$u(x,y) = \sum_{n=1}^{\infty} a_n \cos nx \sinh ny$$

Example 2: Solve the Laplace's equation by the method of separation of variables over $0 \le x \le \pi$, $y > 0$ with $u(x,0) = u(0,y) = u_x(\pi,y) = 0$

Solution : Assume the solution is $u(x,y) = F(x)G(y)$ *then* $F''G + FG'' = 0$

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda^2$$

$$\frac{F''}{F} = -\lambda^2 \quad \Rightarrow \quad F'' + \lambda^2 F = 0$$

$$F = A\sin\lambda x + B\cos\lambda x$$

$$u(0,y) = 0 \quad \Rightarrow \quad F(0) = 0$$

$$0 = A\sin(0) + B\cos(0) \quad \Rightarrow \quad B = 0$$

$$\therefore \quad F = A\sin\lambda x$$

$$u_x(\pi,y) = 0 \quad \Rightarrow \quad F'(\pi) = 0$$

$$F' = A\lambda\cos\lambda x \quad \Rightarrow \quad 0 = A\lambda\cos\lambda\pi \quad \Rightarrow \quad \cos\lambda\pi = 0$$

$$\lambda\pi = \frac{(2n-1)\pi}{2} \quad \Rightarrow \lambda = \frac{2n-1}{2} \qquad n = 1,2,3,\cdots$$

Nwo $-\frac{G''}{G} = -\lambda^2 \quad \Rightarrow \quad G'' - \lambda^2 G = 0$

$$G = C\sinh\lambda y + D\cosh\lambda y$$

$$u(x,0) = 0 \quad \Rightarrow G(0) = 0 \quad \Rightarrow \quad 0 = C\sinh(0) + D\cosh(0) \quad \Rightarrow D = 0$$

$$\therefore \ G = C\sinh\left(\frac{2n-1}{2}\right)y$$

Then $u(x,y) = A\sin\frac{(2n-1)x}{2} \times C\sinh\frac{(2n-1)y}{2} \qquad n = 1,2,3,\cdots$

$$u(x,y) = \sum_{n=1}^{\infty} a_n \sin\frac{(2n-1)x}{2}\sinh\frac{(2n-1)y}{2}$$

Example 3: Solve the Laplace's equation by the method of separation of variables over $0 \le x \le \pi , y > 0$ with $u_y(x,0) = u_x(0,y) = u(\pi,y) = 0$

Solution: $u(x,y) = F(x)G(y)$ *then* $F''G + FG'' = 0$

$$\frac{F''}{F} = -\frac{G''}{G} = -\lambda^2 \Rightarrow \quad \frac{F''}{F} = -\lambda^2 \Rightarrow \quad F'' + \lambda^2 F = 0$$

$$F = A\sin\lambda x + B\cos\lambda x$$

$$u_x(0,y) = 0 \quad \Rightarrow F'(0) = 0$$

$$F' = \lambda A\cos\lambda x - \lambda B\sin\lambda x$$

$$u_x(0,y) = 0 \Rightarrow F'(0) = 0 \Rightarrow 0 = \lambda A\cos(0) - \lambda B\sin(0) \quad \Rightarrow A = 0$$

So $F = B\cos\lambda x$

$$u(\pi,y) = 0 \Rightarrow F(\pi) = 0 \quad \Rightarrow 0 = B\cos(\lambda\pi)$$

$$\cos(\lambda\pi) = 0 \Rightarrow \qquad \lambda\pi = \frac{(2n-1)\pi}{2} \quad \Rightarrow \lambda = \frac{2n-1}{2} \qquad n = 1,2,3,\cdots$$

So $F = B\cos\dfrac{(2n-1)x}{2}$

Nwo $-\dfrac{G''}{G} = -\lambda^2 \quad \Rightarrow \quad G'' - \lambda^2 G = 0$

$$G = C\sinh\lambda y + D\cosh\lambda y \quad \Rightarrow \qquad G' = \lambda C\cosh\lambda y + \lambda D\sinh\lambda y$$

$$u_y(x,0) = 0 \quad \Rightarrow G'(0) = 0 \quad \Rightarrow 0 = \lambda C\cosh(0) + \lambda D\ \sinh(0) \quad \Rightarrow C = 0$$

$$G = D\cosh\frac{(2n-1)y}{2}$$

Then $u(x,y) = B\cos\dfrac{(2n-1)x}{2} \times D\cosh\dfrac{(2n-1)y}{2}$

$$u(x,y) = \sum_{n=1}^{\infty} a_n \cos\frac{(2n-1)x}{2}\cosh\frac{(2n-1)y}{2}$$

H.W: Apply the method of separation of variables to solve Laplace's equation

1. $u_{xx} + u_{yy} = 0$ with $u(x,0) = u_x(0,y) = u_x(\pi,y) = 0$
2. $u_{xx} + u_{yy} = 0$ with $u_y(x,0) = u(0,y) = u\left(\frac{\pi}{2},y\right) = 0$